Lecture 11. Jan 26

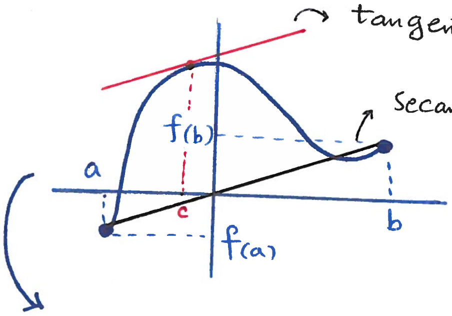
→ First, check worksheet 4 posted on the webpage.

We saw that for a good function in an interval, it's always possible to find a number "c" in the interval such that the tangent line at $x=c$ is parallel to the secant line through the two endpoints.

- What's a good function on an interval?

Take any general function on an interval $[a,b]$.

↷ tangent line at "c".



Secant line through $(a, f(a))$, $(b, f(b))$

tangent line at "c" is parallel to Secant line through the end points.

Good function:

f is continous on $[a,b]$
and
f is differentiable on (a,b)

NO jump, NO hole, NO corner and NO cusp.

$\Rightarrow$ Slopes of the two lines are equal

$$\Rightarrow \left(m_{tan} \text{ at } x=c \right) = m_{sec} \text{ through } (a, f(a)) \text{ and } (b, f(b))$$

$$\Rightarrow f'(c) = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{average slope}}$$

→ This is Mean Value Theorem

Mean Value Theorem (MVT)

Suppose f is a continuous function on the interval $[a,b]$ and f is differentiable on (a,b) then there exists at least a number c between a and b such that
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

You should remember the statement of the Theorem, Conditions and Conclusions. (Be careful about $[a,b]$ and (a,b))

Also, understand the graphical interpretation of the Theorem.

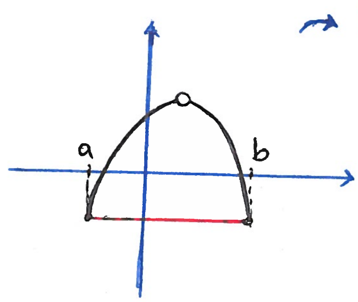
⇝ tangent line at "c" parallel to the secant line.

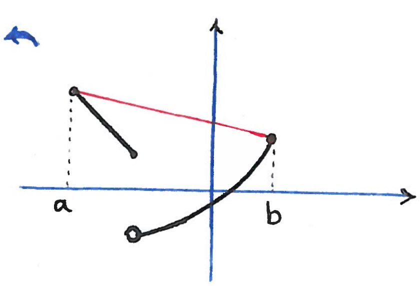
Question: Why the two conditions are important ?

→ Why is continuity of the function on $[a,b]$ needed ?

Consider for example the two functions as follows, where $f(x)$ is NOT Cont's on $[a,b]$ ⇝ there's a hole or jump.

→ MVT fails ←





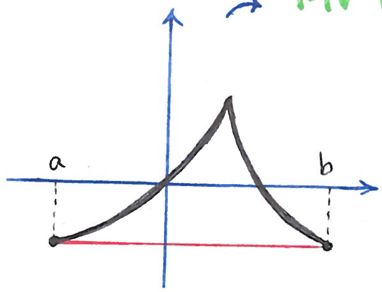
⊛ There's no tangent line parallel to the secant line (red line)

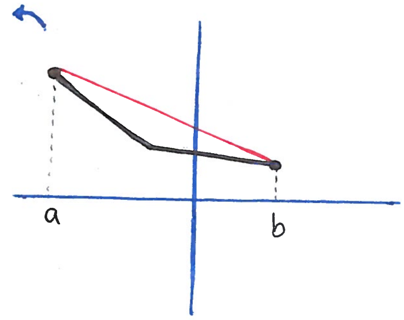
⊛ The discontinuity causes that we can NOT find "c" so that tangent line at "c" is equal to the secant line.

→ Why is differentiability of the function on (a,b) is needed ?

Again Consider two examples of non differentiable functions (Corners or cusps) for which MVT fails.

MVT fails.

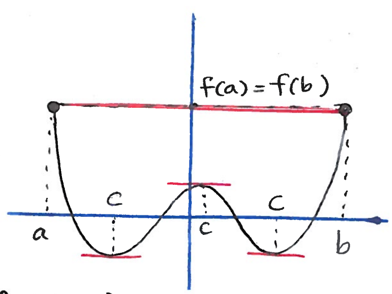




In both cases, there is NO "c" between a and b such that the tangent line at $x=c$ is parallel to the secant line (red line).

A special case of MVT is when we have the additional condition : $f(a) = f(b)$ ⇝ same height for the end points

The conclusion changes to : $f'(c) = \dfrac{f(b) - f(a)}{b-a} = 0$ (equal)



(a special MVT)

↑

Rolle's Theorem.

→ There's at least a "c" such that tangent line is parallel to the secant line, but since $f(a)=f(b)$, secant line is horizontal so the tangent line at c is also horizontal.

> Let f be continuous on $[a,b]$ and differentiable on (a,b). Also, assume $f(a) = f(b)$, then there exists at least a number "c" between a and b such that $f'(c) = 0$.

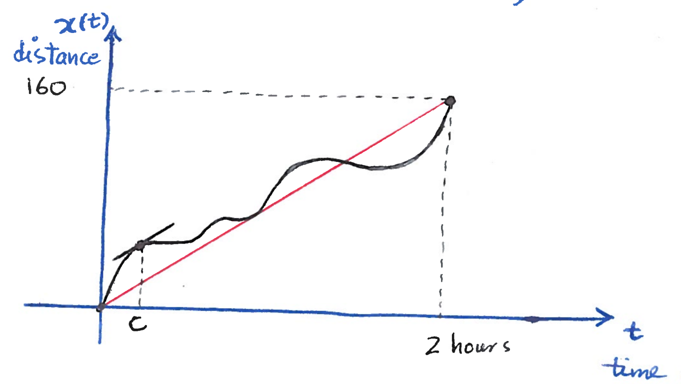
One Important Application of MVT : (Still in use !)

A driver travels a distance of 160 miles on a toll road with a speed limit of 70 miles/hour . The driver completes the 160-mile journey in 2 hours.

At the end of the toll road the driver is issued a speeding ticket . Why ?

(Assume it's a hole-free , cusp-free and smooth road.)

Let's first sketch the trip of this driver.



The road represents a smooth good function. It's continuous on the interval $[0,2]$ (No hole/jump) and differentiable on $(0,2)$ (No corner/cusp). By MVT there's an instant in these 2 hours such that the secant line is parallel to the tangent line at that point. For the distance function, slope of secant line is the average speed and slope of the tangent line is the instantaneous speed.

slope of the secant line = slope of the tangent line at c

average speed in $[0,2]$ = instantaneous speed at instant c

$$\frac{x(2)-x(0)}{2-0} = \frac{160}{2} = \underset{\text{mil/hour}}{80} = V_c$$

At least at one instant "c" the driver has the speed = 80 miles/hour

driver exceeds the speed limit ⟶ TICKET.

Test Yourself.

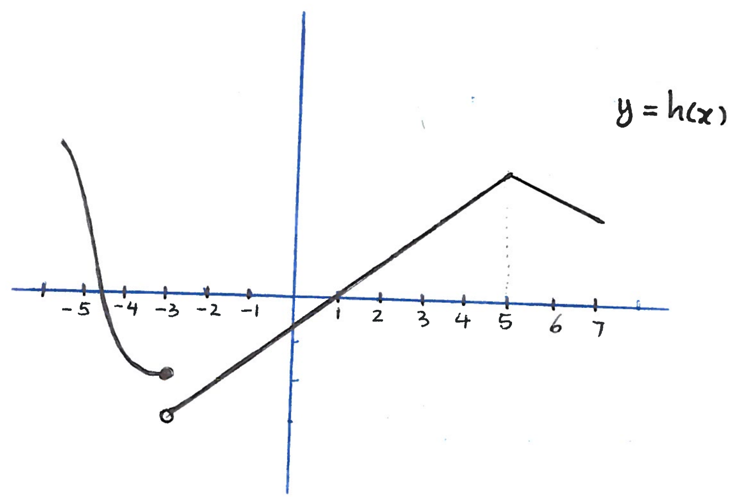
(1) On which of the following intervals MVT applies to the function h over the interval ?

1) $[-5, 1]$

2) $[-1, 3]$

3) $[3, 7]$

4) $[-3, 2]$



(2) The following table gives a few values of function h.

Your friend said that since

x	-3	-2	-1	0
$h(x)$	-6	-1	-4	-5

$$\frac{h(0) - h(-2)}{0 - (-2)} = \frac{-5 - (-1)}{2} = -2$$

So there must be a number "c" in the interval $[-2, 0]$ for which $h'(c) = -2$.

Is this claim True ? If Yes give reasons, if No complete the claim so that it becomes True.

Typical Questions of MVT :

1. Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1, 2]$$

2. Suppose $f(x)$ is continuous and differentiable on $[6, 15]$. Also, we know $f(6) = -2$ and $f'(x) \leq 10$. What's the largest possible value for $f(15)$?

3. Consider the function $f(x) = \dfrac{x}{x+1}$.

a) Find the average slope of the function on the intervals $[0, 1]$ and $[-2, 1]$.

b) In which of these two intervals, can one apply MVT? Explain why?

4. Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has exactly one root.

5. Determine all the numbers c which satisfy the conclusion of Rolle's Theorem. (First, verify whether you can apply Rolle.)

a) $f(x) = x^2 - 5x + 4 \quad [1, 4]$

b) $g(x) = (\sqrt{x-2})(3-x) \quad [2, 3]$

1) $f(x)$ is a polynomial so it's continuous and differentiable everywhere including in $[-1,2]$.

Therefore, by MVT there's a number c between (-1) and 2 such that $f'(c) = \dfrac{f(2)-f(-1)}{2-(-1)}$

We go to the original function to find f', $f(2)$ and $f(-1)$.

$f(x) = x^3 + 2x^2 - x \Rightarrow \begin{cases} f(2) = 2^3 + 2(2)^2 - 2 = 8+8-2 = 14 \\ f(-1) = (-1)^3 + 2(-1)^2 - (-1) = -1+2+1 = 2 \end{cases}$

$\hookrightarrow \dfrac{f(2)-f(-1)}{2-(-1)} = \dfrac{14-2}{3} = \dfrac{12}{3} = 4$

and

$f'(x) = 3x^2 + 4x - 1$

$\Rightarrow f'(c) = 3c^2 + 4c - 1$

$\Rightarrow 3c^2 + 4c - 1 = 4 \Rightarrow 3c^2 + 4c - 5 = 0$

We need to solve the quadratic for c :

Use quadratic formula:

if $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

So $3c^2 + 4c - 5 = 0 \Rightarrow c = \dfrac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2 \times 3}$

$\Rightarrow c = \dfrac{-4 \pm \sqrt{16+60}}{6} \Rightarrow$ $c = \dfrac{-4+\sqrt{76}}{6}$

$c = \dfrac{-4-\sqrt{76}}{c}$

2) f cont's and diff'able in $[6,15]$

$f(6) = -2$ and $f'(x) \leq 10$.

By MVT : there's c $\quad f'(c) = \dfrac{f(15) - f(6)}{15 - 6} \Rightarrow f'(c) = \dfrac{f(15) - (-2)}{9}$

cross multiply $\rightsquigarrow$ $9f'(c) = f(15) + 2$

isolate $f(15)$ $\rightsquigarrow$ $f(15) = 9f'(c) - 2$

for all x, we have $f'(x) \leq 10$ $\rightsquigarrow$

$$\begin{aligned} f(15) &= 9f'(c) - 2 \\ &\leq 9 \times 10 - 2 \\ &\leq 90 - 2 = 88 \end{aligned}$$

$\Rightarrow \boxed{f(15) \leq 88}$ The largest possible value for $f(15)$ is 88.

3) average slope of f on $[a,b]$: $\dfrac{f(b) - f(a)}{b - a}$, $\quad f(x) = \dfrac{x}{x+1}$

(a)

$\rightarrow$ on $[0,1]$: $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{\frac{1}{2} - 0}{1} = \dfrac{1}{2}$

$f(0) = \dfrac{0}{0+1} = 0$, $\quad f(1) = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\rightarrow$ on $[-2,1]$: $\dfrac{f(1) - f(-2)}{1 - (-2)} = \dfrac{\frac{1}{2} - 2}{3} = \dfrac{-\frac{3}{2}}{3} = -\dfrac{3}{2} \times \dfrac{1}{3} = -\dfrac{1}{2}$

$f(-2) = \dfrac{-2}{-2+1} = \dfrac{-2}{-1} = 2$

(b) f is NOT continuous at $x = -1$ because the denominator becomes zero, so any interval that contains this discontinuity can NOT be used for MVT $\rightsquigarrow$ In this case : in $[-2,1]$ MVT may fail.