Definition :

Increasing Function :

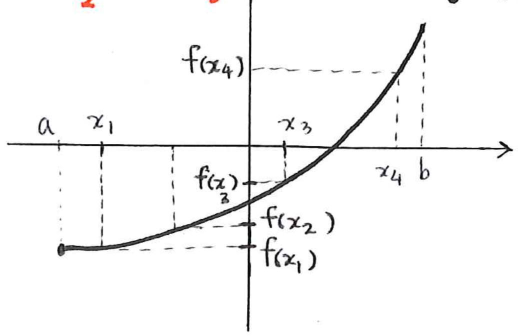
As we move from left to right on the x-axis, the function values go higher and higher

$\Downarrow$

As x increases, $f(x)$ increases.

translate into math notation

$f(x_1) < f(x_2) < f(x_3) < \cdots$



Formal Definition : A function f is increasing on the interval (a,b) if $a < x_1 < x_2 < b$ implies that $f(x_1) < f(x_2)$.

(or $f(x_1) \leq f(x_2)$)

$$\boxed{x_1 < x_2 \Rightarrow f(x_1) < f(x_2)}$$

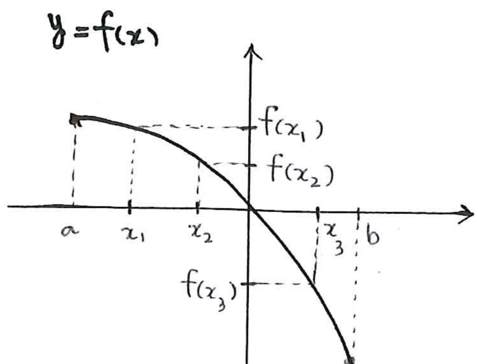
Decreasing Function :

As we move from left to right on the x-axis, the function goes downward.

$\Downarrow$

As x increases, $f(x)$ decreases.

translate



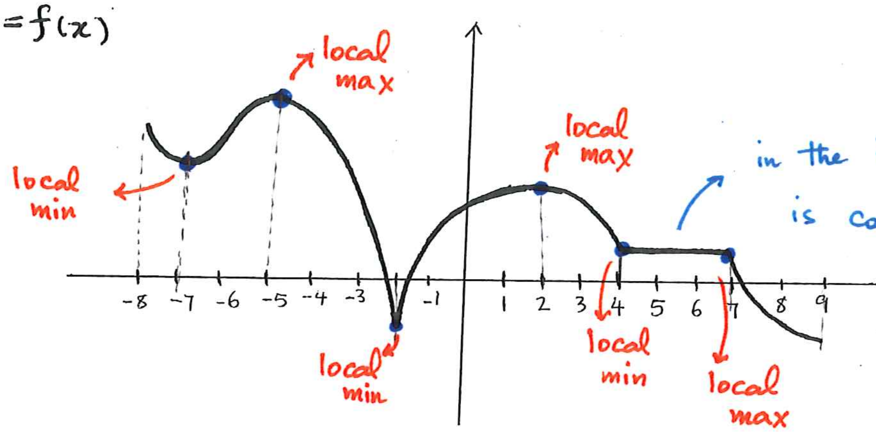
$f(x_1) > f(x_2) > f(x_3) > \cdots$

Formal Definition : A function f is decreasing on (a,b) if $a < x_1 < x_2 < b$ implies that $f(x_1) > f(x_2)$ (or $f(x_1) \geq f(x_2)$)

$$\boxed{x_1 < x_2 \Rightarrow f(x_1) > f(x_2)}$$

Example · List the (longest) intervals on which the following function is increasing or decreasing.

$y = f(x)$



in the interval $[4,7]$ the function is constant. It is both increasing and decreasing.

increasing on: $[-7,-5]$, $[-2,2]$

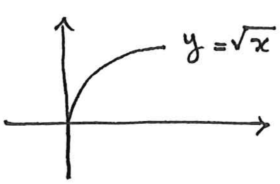
decreasing on: $[-8,-7]$, $[-5,-2]$, $[2,4]$, $[7,9]$

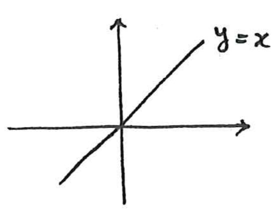
Exercise. List the critical numbers for the above function. Which of them are local min? Which are local max?

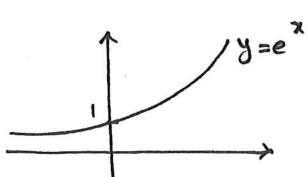
Critical numbers: $x = -7, -5, -2, 2, 4, 7$ and all the numbers in the interval $(4,7)$.

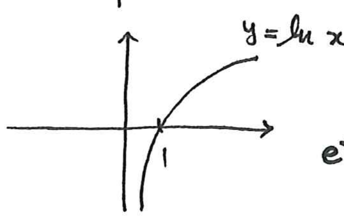
Question. Among the well-known functions that you've learned determine the increasing and decreasing ones.

Increasing functions:



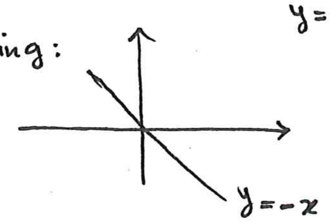


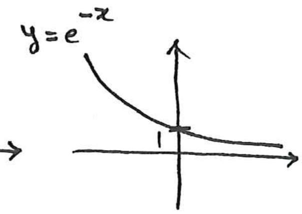




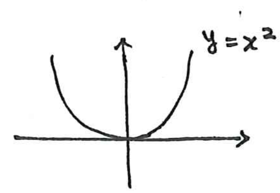
etc ...

decreasing:





parabola:



$(-\infty, 0]$ decreasing

$[0, +\infty]$ increasing

Relationship between f being increasing/decreasing $\longleftrightarrow$ f' being $+/-$

$\rightarrow$ When there's No graph, we need to use f' to find about f being increasing or decreasing.

$\longrightarrow$ 1st direction

Assume f is a differentiable function on the interval (a,b).

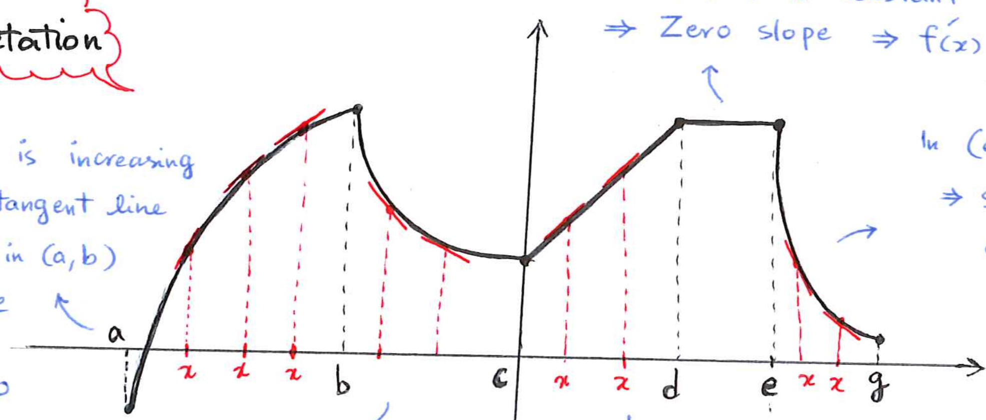
(I) If f is increasing on (a,b) then $f'(x) > 0$ for all x in the interval (a,b).

(II) If f is decreasing on (a,b) then $f'(x) < 0$ for all x in the interval (a,b).

(III) If f is constant on (a,b) then $f'(x) = 0$ for all x in (a,b).

Graphical Interpretation

In (d,e) f is constant $\Rightarrow$ Zero slope $\Rightarrow f'(x) = 0$



In (a,b) f is increasing $\Rightarrow$ slope of tangent line at any x in (a,b) is positive

$\Rightarrow f'(x) > 0$

In (e,g) f is decreasing $\Rightarrow$ slope of the tangent line at any x in (e,g) is negative

$\Rightarrow f'(x) < 0$

in (b,c) f is decreasing $\Rightarrow$ slope of the tangent line at any x in (b,c) is negative

$\Rightarrow f'(x) < 0$

In (c,d) f is increasing $\Rightarrow$ slope of the tangent line at any x in (c,d) is positive

$\Rightarrow f'(x) > 0$

Exercise. Use the definition of an increasing/decreasing function to show that the above statement is always true. (Without graphing, use formulas to show that.)

Hint: Use the limit definition of the derivative.

Relationship between f and f'

← Opposite direction

Assume f is a differentiable function on the interval (a,b)

(I) If $f'(x) > 0$ for all x in (a,b) then f is increasing in (a,b).

(II) If $f'(x) < 0$ for all x in (a,b) then f is decreasing in (a,b).

(III) If $f'(x) = 0$ for all x in (a,b) then f is constant in (a,b).

Question: Use the given info about the sign of derivative to show that f is increasing on (a,b).

i.e. Assume f' is positive we show that for $a < x_1 < x_2 < b$ we have $f(x_1) < f(x_2)$.

Solution. Take the interval (a,b) and choose two arbitrary numbers x_1 and x_2 in this interval such that $a < x_1 < x_2 < b$, it looks like a —— x_1 —— x_2 —— b. By the assumption, f is differentiable on (a,b), so it's continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Therefore, we are allowed to use MVT on (a,b). By MVT, there is a number c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If we show that $f(x_2) - f(x_1)$ is positive then this means that $f(x_2) > f(x_1)$ and we are done. We have

$$\underbrace{f'(c)}_{+} = \frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{+}} \longrightarrow \text{So } f(x_2) - f(x_1) > 0 \Rightarrow \boxed{f(x_2) > f(x_1)}$$

By the given condition we know for all x in (a,b) $f'(x) > 0$ including $f'(c)$.

x_2 is larger than x_1 (to the right of x_1)

increasing