Solutions to Practice Problem of Lecture 3.

1) $xy + x^2 + y^2 = x$

a) $\underbrace{\text{Slope of the tangent line}}_{y' = \frac{dy}{dx}}$ at the $\underbrace{x\text{-int}}_{y=0}$ with positive x-coordinate.

First we use implicit differentiation to find y':

$$\underbrace{(1 \cdot y + xy')}_{\text{product rule for } xy} + 2x + 2yy' = 1$$

$$\xrightarrow[\text{and isolate } y']{\text{simplify}} \quad y + xy' + 2x + 2yy' = 1$$

$$y'(x+2y) = 1 - y - 2x$$

$$\xrightarrow[\text{for } y']{\text{solve it}} \quad y' = \frac{1-y-2x}{x+2y}$$

Now we need to find the point, both its x and y coordinate.

The point is x-intercept $\longrightarrow y = 0$

$$\xrightarrow[\text{plug } y=0 \text{ to find } x]{\text{Now go to the equation}} \quad x \cdot 0 + x^2 + 0^2 = x$$

$$\xrightarrow{\text{simplify}} \quad x^2 - x = 0$$

$$\xrightarrow{\text{factor}} \quad x(x-1) = 0$$

$$\longrightarrow \quad x = 0 \quad \text{or} \quad \boxed{x = 1}$$

This is the x-coordinate we should choose, because it's the one which is positive.

Point : $(1, 0)$

Evaluate : $y' = \dfrac{1-0-2}{1+0} = \dfrac{-1}{1} = -1 \implies \boxed{m_{tan} \text{ at } (1,0) = -1}$

(b) We have $y' = \dfrac{1-y-2x}{x+2y}$, we apply quotient rule and implicit diff and find the derivatives of both sides.

$$(y')' = \left(\frac{1-y-2x}{x+2y}\right)'$$

Recall : $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ → Quotient Rule

$$y'' = \frac{\overbrace{(-y'-2)}^{f'}\overbrace{(x+2y)}^{g}-\overbrace{(1-y-2x)}^{f}\overbrace{(1+2y')}^{g}}{(x+2y)^2}$$

$$y'' = \frac{-xy' - \cancel{2yy'} - \cancel{2x} - 4y - 1 - 2y' + y + \cancel{2yy'} + \cancel{2x} + 4xy'}{(x+2y)^2}$$

$$y'' = \frac{3xy' - 2y' - 4y - 1}{(x+2y)^2} \xrightarrow[y']{\text{substitute}} y'' = \frac{(3x-2)\frac{1-2x-y}{x+2y} - 4y - 1}{(x+2y)^2}$$

$$\xrightarrow[y'=-1]{x=1,\, y=0} \frac{d^2y}{dx^2} = \frac{3\cdot(-1) - 2(-1) - 0 - 1}{(1+0)^2} = \boxed{\frac{-3+2-1}{1} = -2}$$

2) $x^2 \underbrace{\tan}_{\text{out}}\left(\underbrace{\tfrac{\pi}{4}y}_{\text{in}}\right) + 2x \underbrace{\ln}_{\text{out}} \underbrace{y}_{\text{in}} = 16$

$$\xrightarrow[\text{derivative of both sides.}]{\text{take the}} \underbrace{\left(2x \tan\left(\tfrac{\pi}{4}y\right) + x^2 \overbrace{\left(\underbrace{1+\tan^2\left(\tfrac{\pi}{4}y\right)}_{\text{outside derivative}}\right)}^{\text{derivate of tan}} \cdot \underbrace{\tfrac{\pi}{4}y'}_{\text{inside derivative}}\right)}_{\text{product rule for the 1st term}} + \underbrace{\left(2\ln y + 2x \underbrace{\tfrac{1}{y}}_{\ln \text{ derivative}} \cdot \underbrace{y'}_{\text{inside derivative}}\right)}_{\text{product rule for the 2nd term}} = 0$$

The question asks for the derivative at a point

with $y=1 \xrightarrow[\text{to find } x]{\text{go to the equation}} x^2 \underbrace{\cancel{\tan\left(\tfrac{\pi}{4}\right)}}_{1} + 2x \underbrace{\cancel{\ln(1)}}_{0} = 16 \Rightarrow x^2 = 16$

$\Rightarrow x = 4 \quad , \quad x = -4$

$\xRightarrow{\text{Points}} (4,1) \quad , \quad (-4,1)$

We just need to evaluate y' at the points, so there's no need to isolate and solve for y', first plug in the point then solve for y':

$$(4,1) \xrightarrow[\text{derivative above}]{\text{Go to the long}} 2\cdot 4 \underbrace{\cancel{\tan\left(\tfrac{\pi}{4}\right)}}_{1} + 4^2\left(1 + \underbrace{\cancel{\tan^2\left(\tfrac{\pi}{4}\right)}}_{1}\right)\cdot \tfrac{\pi}{4} y' + \left(2 \underbrace{\cancel{\ln 1}}_{0} + 2\cdot 4 \cdot \tfrac{1}{1} \cdot y'\right) = 0$$

$$8 + \underbrace{32\tfrac{\pi}{4}}_{8\pi} y' + 8y' = 0 \xRightarrow[y']{\text{isolate}} y'(8\pi + 8) = -8$$

$$\xrightarrow{\text{solve}} y' = \frac{-8}{8\pi+8} = \frac{-8}{8(\pi+1)} = \boxed{\frac{-1}{\pi+1}}$$

3) $y^2+4xy-2x^2=3$

a) Horizontal tangent line $\rightsquigarrow$ $y'=0$

b) Vertical " " $\rightsquigarrow$ y' NOT defined.

First, we use implicit differentiation to find y'.

$$2yy'+4(1.y+xy')-2.2x=0$$

$$\xrightarrow{\text{simplify}} 2yy'+4y+4xy'-4x=0$$

$$\xrightarrow[\text{\& isolate } y']{\text{factor}} y'(2y+4x)=4x-4y$$

$$\xrightarrow[y']{\text{solve for}} y'=\frac{4x-4y}{2y+4x}$$

$\xrightarrow{y'=0}$ numerator $=0 \Rightarrow 4x-4y=0 \Rightarrow \boxed{x=y}$

$\xrightarrow[\text{NOT defined}]{y'}$ denominator $=0 \Rightarrow 2y=-4x \Rightarrow \boxed{y=-2x}$

We see that when setting $y'=0$ we find a relation between x and y rather than a point i.e. $x=y$. We plug this into the original equation of the curve to make it a function of only x:

$$y^2+4xy-2x^2=3 \xrightarrow{x=y} x^2+4x.x-2x^2=3$$

$$\xrightarrow{\text{Simplify}} 3x^2=3$$

$$\longrightarrow x^2=1$$

$$\Rightarrow x=1 \quad , \quad x=-1$$

$$\xrightarrow{y=x} y=1 \quad , \quad y=-1$$

$\Longrightarrow$ at $(1,1)$ and $(-1,-1)$ the tangent line is horizontal.

We do similarly for vertical tangent line:

$$y^2+4xy-2x^2=3 \xrightarrow{y=-2x} (-2x)^2+4x(-2x)-2x^2=3$$

$$\longrightarrow 4x^2-8x^2-2x^2=3$$

$$\Rightarrow -6x^2=3$$

$$\Rightarrow x^2=-\frac{1}{2} \quad \text{Never possible}$$

$\Rightarrow$ At NO point the tangent line is vertical.