Important Notes: ① Make sure you have the correct sign for rates of change ⇝ decrease ⊖, increase ⊕

② Do NOT substitute the given values for changing quantities before differentiating.

WORKSHEET 2: Related Rates

MATH 110, Jan 15, 2018

⋆ Include the units in your answer.

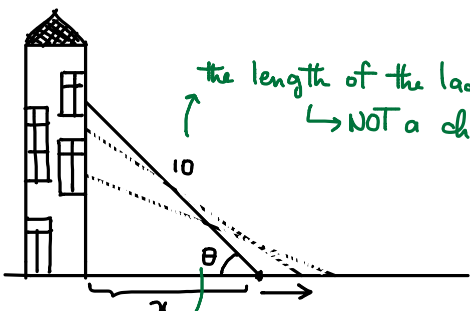
1. A ladder 10 feet long leans against a building. If the base of the ladder slides away from the building horizontally at a rate of 4 ft/sec. At what rate is the angle between the ladder and the ground is changing when the base is 8 feet from the building?

$\frac{dx}{dt} = x'(t) = 4$

$\frac{d\theta}{dt} = \theta'(t)$ ⇝ unknown

At this moment when $x = 8$

↳ Note: Although we know $x = 8$ at this moment we substitute it only after differentiating.



the length of the ladder is fixed ↳ NOT a changing quantity

As the ladder slides away x and θ are changing ⇝ Changing quantities

Solution: How to relate the variables x and θ?

We have an angle and a right triangle ⇝ Use trigs to relate

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenus}} \xrightarrow[\text{on } t]{\text{depend}} \cos\theta(t) = \frac{x(t)}{10}$ (out: cos, in: $\theta(t)$)

$\xrightarrow[\text{respect to time}]{\text{differentiate with}} -\sin\theta(t)\cdot\theta'(t) = \frac{1}{10}x'(t)$

$\sin\theta = \frac{\text{opposite}}{\text{hypotenus}} = \frac{6}{10}$

$\xrightarrow{x=8}$ opposite $= \sqrt{10^2-8^2} = \sqrt{36} = 6$

$\xrightarrow[\sin\theta(t)]{\text{Evaluate}} -\frac{6}{10}\cdot\theta'(t) = \frac{1}{10}\cdot 4 \Rightarrow \boxed{\theta'(t) = -\frac{4\times 10}{6\times 10} = -\frac{2}{3}\ \text{rad}/s}$

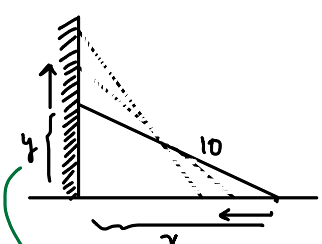
2. A ladder 10 feet long leans against a wall. If the base of the ladder is being pushed toward the wall horizontally at a rate of 4 ft/sec, how fast is the top of the ladder moving up the wall when the top of the ladder is 8 feet from the ground.

$\frac{dx}{dt} = x'(t) = -4$

At this moment $y = 8$

$\frac{dy}{dt} = y'(t)$ ⇝ unknown

Note: Again we substitute this quantity after deriving because y is changing.



The ladder goes toward wall
↳ x decreases
↳ Negative rate of change

Top of the ladder goes upward
↳ y increases
↳ positive rate of change

Solution: How to relate x and y? Pythagorean Theorem

$x^2 + y^2 = 10^2$

$\xrightarrow{\text{functions of } t} (x(t))^2 + (y(t))^2 = 10^2$

$\xrightarrow[\text{with respect to time}]{\text{differentiate}} 2\underbrace{x}_{}\underbrace{x'}_{-4} + 2\underbrace{y}_{8}\underbrace{y'}_{\text{unknown}} = 0$

$x = \sqrt{10^2-8^2} = \sqrt{36} = 6$

$\xrightarrow[\text{Pythagorean for } x]{\text{Use again}} 2\cdot 6\cdot(-4) + 2\cdot 8\cdot y' = 0$

$\xrightarrow[\text{for } y']{\text{solve}} -48 + 16y' = 0 \Rightarrow \boxed{y' = \frac{48}{16} = 3\ ft/s}$

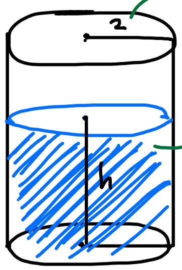
always fixed

$\frac{dV}{dt} = V'(t)$

3. A cylindrical tank with radius 2 meters is being filled up with water at a rate of 3 m^3/min. How fast is the height of the water increasing when the depth of the water in the tank is 4 meters.

$dh/dt = h'(t)$ ⇝ unknown

at this moment when $h=4$ ↓ sub after differentiating



radius is fixed for all heights of the water.

tank is filled in ⇝ (V) Volume changes ↳ increase ↳ positive rate

↓ (h) height increases ↓ positive rate

Solution : $V = \pi r^2 h$, $r = 2$

$\xrightarrow[\text{of time}]{\text{functions}}$ $V(t) = 4\pi h(t)$

$\xrightarrow{\text{Derive}}$ $V'(t) = 4\pi h'(t)$

$\xrightarrow{\text{sub}}$ $3 = 4\pi h'$

$\Rightarrow$ $\boxed{h' = \frac{3}{4\pi}\ m/\text{min}}$

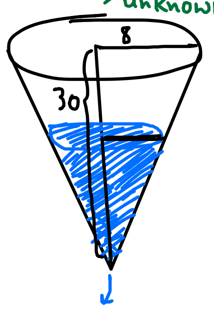
Note : We didn't use $h=4$
it sometimes happens that we have extra info !

4. A funnel in the shape of an inverted cone is 30 cm deep and has a diameter across the top of 16 cm. Liquid is flowing out of the funnel at the rate of 12 cm^3/sec. At what rate is the height of the liquid decreasing at the instant when the liquid in the funnel is 20 cm deep?

$\frac{dh}{dt} = h'(t)$ ↳ unknown

$\frac{dV}{dt} = V'(t) = -12$

at this moment $h = 20$



liquid flows out ⇝ what quantities are changing over time ?

Volume , height and RADIUS → Unlike Q3 , radius is NOT fixed , it's decreasing.

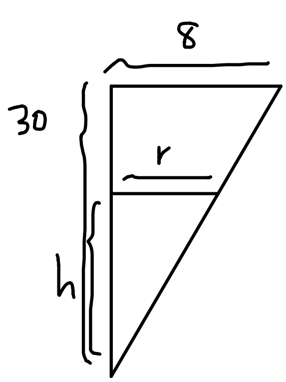
all decreasing

$V = \frac{1}{3}\pi r^2 h$

$\xrightarrow[\text{changing}]{\text{all are}}$ $V(t) = \frac{1}{3}\pi\, r(t)^2\, h(t)$

$\xrightarrow[\text{product rule}]{\text{Differentiate by}}$ $V'(t) = \frac{1}{3}\pi\left(2rr'h + r^2 h'\right)$

we don't have any info about rate of change of radius
↳ We need to get rid of r
$\xrightarrow{\text{how?}}$ Use similar triangles



30 and 8 are fixed,
h and r are changing over time

$\frac{h}{30} = \frac{r}{8} \Rightarrow 8h = 30r \Rightarrow r = \frac{8}{30}h \Rightarrow \boxed{r = \frac{4}{15}h}$

all the given info is about h , so we write r in terms of h and plug it back into the volume formula.

Go back to V formula:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{4}{15}h\right)^2 h$$

$$\Longrightarrow V(t) = \frac{\pi}{3}\cdot\frac{16}{225}\, h^3(t)$$

$$\xrightarrow[\text{differentiate}]{\text{Now}} V'(t) = \frac{\pi}{\cancel{3}}\cdot\frac{16}{225}\,\cancel{3}\, h^2(t)\, h'(t) \to \text{unknown}$$

$$\xrightarrow[\text{info}]{\text{sub the}} -12 = \frac{\pi\cdot 16}{225}\cdot(20)^2\, h'(t)$$

$$\xrightarrow{\text{solve}} \boxed{h'(t) = \frac{-12}{400}\cdot\frac{225}{16\pi} \quad \text{cm/sec}}$$