

Name (print):

Student number:

Section (*Please circle one*): 001 002 003 004



University of British Columbia

MATH 110 (Section 001): FEBRUARY EXAM

Date: *February 12, 2013*

Number of pages: *9 (including cover page)*

Exam type: *Closed book*

Aids: *No calculators or other electronic aids*

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Question	Points	Score
1	4	
2	4	
3	4	
4	5	
5	6	
6	5	
7	7	
Total:	35	

4 marks

1. Find all local maximum and minimum values for the function $f(x) = \frac{4x}{x^2 + 1}$.

Solution: First, we should check the domain of the function. We see that the denominator is never zero and hence the function is defined for all real numbers. Therefore the domain is all real numbers. To find the extrema values, we need to find the first derivative. Let:

$$\begin{aligned} u(x) &= 4x & u'(x) &= 4 \\ v(x) &= x^2 + 1 & v'(x) &= 2x \end{aligned}$$

Then we get the derivative to be:

$$\begin{aligned} f'(x) &= \frac{4(x^2 + 1) - (4x)(2x)}{(x^2 + 1)^2} \\ &= \frac{4(1 - x^2)}{(x^2 + 1)^2} \end{aligned}$$

So the critical points happen when the numerator is zero ($f'(x) = 0$) or when the denominator is zero ($f'(x)$ undefined). Since the denominator is never zero, so all we have to do is check the numerator.

$$4(1 - x^2) = 0 \qquad x = \pm 1$$

So the critical points are at $x = \pm 1$. We know from class that the local extrema must happen at critical points, so now all we have to do is apply the first derivative test to check the increasing/decreasing over the intervals.

Interval	$f'(x)$	INC/DEC
$(-\infty, -1)$	$f'(-2) < 0$	DEC
$(-1, 1)$	$f'(0) > 0$	INC
$(1, \infty)$	$f'(2) < 0$	DEC

Since the derivative goes from DEC to INC over $x = -1$, it must be a local min. Similarly, since the derivative goes from INC to DEC over $x = 1$, it must be a local max. The actual points are $(-1, f(-1)) = (-1, -2)$ and $(1, f(1)) = (1, 2)$.

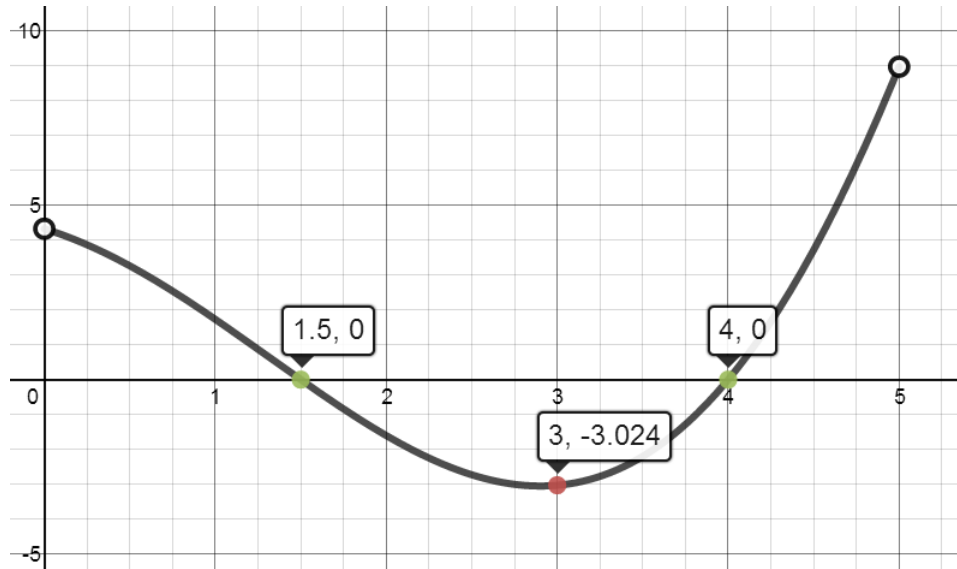
Solution: Using the second derivative test, we need to find the second derivative:

$$f''(x) = \frac{8x(x^2 - 3)}{(x^2 + 1)^3}.$$

Then we just substitute the critical points to get $f''(-1) > 0$ (CU) and $f''(1) < 0$ (CD). The second derivative test tells us that $x = -1$ must be a local min (critical point and CU) and $x = 1$ must be a local max (critical points and CD).

4 marks

2. Below is a sketch of the graph of $f'(x)$ (note that it is the graph of the *derivative* of $f(x)$). Clearly label every x value at which $f(x)$ has a local maximum, local minimum or inflection point. Below the picture, justify each of your labels in one or two sentences.

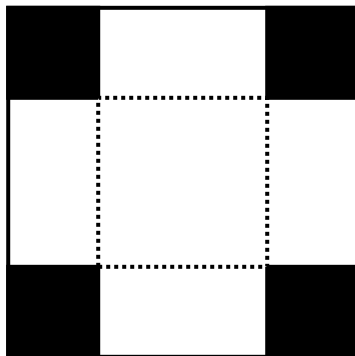
Solution:

The three points are marked on the plot above. I claim that $x = 1.5, 4$ are critical points and $x = 3$ is an inflection point.

1. The point $x = 1.5$ is a local maximum since it is a critical point where the derivative goes from positive (increasing $f(x)$) to negative (decreasing $f(x)$). So by the first derivative test, we have that $x = 1.5$ is a local maximum.
2. The point $x = 4$ is a local minimum since it is a critical point where the derivative goes from negative (decreasing $f(x)$) to positive (increasing $f(x)$). So by the first derivative test, we have that $x = 4$ is a local minimum.
3. The point $x = 3$ is an inflection point since it is a critical point of $f'(x)$ which implies $f''(3) = 0$. So $x = 3$ is a point of interest for the function $f(x)$. Since $f'(x)$ goes from DEC to INC around $x = 3$, we know that $f''(x)$ goes from negative to positive and hence $x = 3$ must be an inflection point since there is a change in concavity.

4 marks

3. Suppose you are asked to construct a box out of a sheet of paper measuring 8.5 inches by 8.5 inches by cutting off four corners of the sheet and folding along the dotted lines, as in the diagram below.



If the four corner squares have a side length of x inches, the resulting box has a volume of

$$V(x) = x(8.5 - 2x)^2$$

cubic inches. The derivative may be taken to be

$$V'(x) = (8.5 - 2x)(8.5 - 6x).$$

Explain carefully how you may conclude that in order to construct a box of maximal volume, you should cut off four corner squares of side length $x = \frac{8.5}{6}$ inches.

Solution: Preamble: We first set up the equation by realising the length and width of the base is whatever is left after we cut two squares of side x from the full 8.5 inches. The height of the box will then be x . So the volume of the box is given by $V(x) = x(8.5 - 2x)^2$. We then look for the maximum by hunting for the critical points via the first derivatives. Using the product and chain rule to get $V'(x) = (8.5 - 2x)(8.5 - 6x)$ as given. This gives the critical points as $x = \frac{8.5}{2}, \frac{8.5}{6}$. **Solution:** We then classify the critical points by partitioning the domain of the function. Since the smallest square we can cut is $x = 0$, and the largest is $x = \frac{8.5}{2}$, we have that domain being $x \in [0, \frac{8.5}{2}]$. Since $V(x)$ is a polynomial, it is continuous and differentiable on the closed domain. Hence we can use the closed interval method to find the maximum.

$$\begin{aligned} V(0) &= 0 \\ V\left(\frac{8.5}{6}\right) &= \frac{8.5}{6} \left(8.5 - 2 \cdot \frac{8.5}{6}\right)^2 > 0 \\ V\left(\frac{8.5}{2}\right) &= 0 \end{aligned}$$

The largest value obtained is the only positive one obtained, which is at $x = \frac{8.5}{6}$, and hence it must be the maximum.

Solution: Alternative solution: If we instead argue that the domain should be $(0, \frac{8.5}{2})$ because we cannot cut 0 and cannot cut the entire thing, then the only critical point in the interval is $\frac{8.5}{6}$. We then have to use the first or second derivative test to justify that it is a maximum.

1. Using the first derivative test: If $x < \frac{8.5}{6}$, then $(8.5 - 2x) > 0$ and $(8.5 - 6x) > 0$, so $V'(x) > 0$. Similarly, if $x > \frac{8.5}{6}$, then $(8.5 - 2x) > 0$ and $(8.5 - 6x) < 0$, so $V'(x) < 0$. That means $V'(x)$ goes from INC to DEC and hence $x = \frac{8.5}{6}$ must be a maximum.
2. Using the second derivative test: We need to find the derivative $V''(x) = 24x - 68$. Then substitute in the point we get $V''(\frac{8.5}{6}) < 0$ and hence must be a maximum.

Finally, we have to explain why the maximum we found is a global maximum and not just a local maximum. The simplest way I think is to say that the function is increasing to the left of the critical points and so *every* point in the domain to the left has a lower value than the critical point. Similarly, the function is decreasing to the right and hence *every* point in the domain to the right has a lower value than the critical point. So that means the critical point is the highest point in the domain, the absolute maximum. **This last part is important since we want to show a global max and not simply a local max.**

4. Let $f(x) = \sin(2x) + x$.

4 marks

(a) Determine where $f(x)$ is concave up on the interval $[0, \pi]$.

Solution: To find where the function is concave up, we need to find the second derivative of the function.

$$\begin{aligned} f(x) &= \sin(2x) + x \\ f'(x) &= 2\cos(2x) + 1 \\ f''(x) &= -4\sin(2x) \end{aligned}$$

To find the interval of concave up, we look for points of interest. Knowing that $\sin(2x)$ is defined everywhere, we only need to find points where $-\sin(2x) = 0$. We know $\sin(u) = 0$, when $u = 0, \pm\pi, \pm2\pi, \pm3\pi \dots$. Checking each of those, we find that only $u = 0, \pi, 2\pi$ are in the required interval, which translates to $x = 0, \frac{\pi}{2}, \pi$. Looking at the intervals:

Interval	$f''(x)$	CU/CD
$(0, \frac{\pi}{2})$	$f''(\frac{\pi}{4}) = -4 < 0$	CD
$(\frac{\pi}{2}, \pi)$	$f''(\frac{3\pi}{4}) = 4 > 0$	CU

So the interval of concave up is given by: $(\frac{\pi}{2}, \pi)$.

1 mark

(b) Describe where $f(x)$ is concave up on the interval $(-\infty, \infty)$.

Solution: We recognise that $\sin(2x)$ is periodic π (it is $\sin(x)$ compressed by a factor of two and hence has half the normal period of $\sin(x)$). So whatever is true for on the interval $(0, \pi)$ must be true for every period. So the interval of concave up must be $(\frac{\pi}{2} + k\pi, \pi + k\pi)$ for any integer k .

5. Let $f(x) = e^x + 2x^3$.

2 marks

- (a) Use the Intermediate Value Theorem to show that $f(x)$ has at least one root.

Solution: We have to check that the function is continuous and differentiable over the real numbers. This can be as simple as realising that $f(x)$ is the sum of two differentiable and continuous functions and so the result, $f(x)$, is also differentiable and continuous. Now we just need a point above and below zero. I go overboard and choose:

$$f(-1000) < 0 \qquad f(1000) > 0$$

The first inequality comes from e^{-1000} is really small and positive while the second term is large and negative. The second inequality comes from each of the terms being positive. So then by the IVT, there is a point c in the interval $(-1000, 1000)$ where $f(c) = 0$, and the root we are looking for.

2 marks

- (b) Explain why $f(x)$ is increasing on the interval $(-\infty, \infty)$.

Solution: To check whether a function is increase, we look at the first derivative: $f'(x) = e^x + 6x^2$. Now we look for the derivative is zero or undefined. each term in the derivative is defined over the real numbers, it is defined everywhere. We simply look for when it is zero. We note that e^x is always positive and $6x^2 \geq 0$. So that means the sum of the two must always be positive. Hence $f'(x) > 0$ for all x and $f(x)$ must always be increasing.

2 marks

- (c) Use the Mean Value Theorem or Rolle's Theorem, along with your answer from part (b), to show that $f(x)$ cannot have two roots. (Hint: If there are two roots, what might go wrong?)

Solution: Hint: Let's use the hint. Suppose there are two roots. Lets say they're at $x = a, b$ with $a < b$. Since $f(x)$ is continuous and differentiable over the real numbers (from the first part), it must be continuous over $[a, b]$ and differentiable over (a, b) . Further, we have $f(a) = f(b) = 0$. So by Rolle's theorem (or MVT), there must be a point c in the interval (a, b) such that $f'(c) = 0$. However, we saw from the previous part that the derivative is always positive and therefore cannot be zero. So that means our assumption of having two roots must be flawed. Therefore, $f(x)$ cannot have two roots

Solution: Alternative solution: We can also use the contrapositive of Rolle's Theorem. If $f(x)$ is continuous on $[a, b]$, differentiable on (a, b) , and $f'(c) \neq 0$ for all c in (a, b) , then $f(a) \neq f(b)$. If $x = a$ is root and $x = b$ is any other point in the domain, we know $f(b) \neq 0$ since $f'(x) \neq 0$ for every point in the interval (regardless of what b I picked).

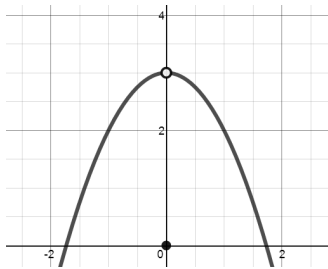
6. Complete the answers to the following true or false questions by providing a counterexample or explanation as requested. (In the case of providing a counterexample, a picture suffices.)

1 mark

- (a) The following statement is false. Provide a counterexample demonstrating that it is false.

If a function is defined everywhere, increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, then the function has a global maximum at 0.

Solution: An example:



Examples of algebraic functions: (Punctured parabola)

$$f(x) = \begin{cases} -x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

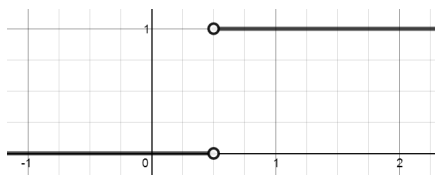
The function is increasing and the decreasing, but a hole is punched where the max is supposed to be. So no global maximum.

1 mark

- (b) The following statement is false. Provide a counterexample demonstrating that it is false.

If $f(0) = 0$ and $f(1) = 1$, then there is a number c in the interval $(0, 1)$ such that $f'(c) = 1$.

Solution: An example:



Examples of algebraic functions: (A shift of $\frac{|x|}{x}$).

$$f(x) = \frac{|x - 0.5|}{2(x - 0.5)} + 0.5$$

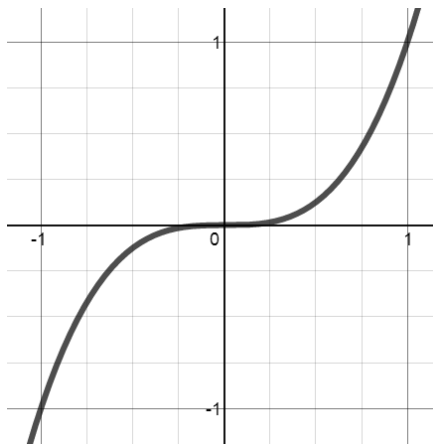
The function has zero slope everywhere there is a tangent line and an undefined tangent line at $x = 0.5$. No slope of 1.

1 mark

- (c) The following statement is false. Provide a counterexample demonstrating that it is false.

A critical point can never be an inflection point.

Solution: An example:



Examples of algebraic functions: (A cubic).

$$f(x) = x^3$$

The derivative $f'(x) = 3x^2$ and so critical points at $x = 0$. The second derivative $f''(x) = 6x$ with a point of interest at $x = 0$. Since $f''(x)$ is negative on one side and positive on the other, there is a change in concavity and hence $x = 0$ is also a inflection part.

2 marks

- (d) The following statement is true. Explain why.

The sum of two decreasing functions is a decreasing function.

Solution: Consider two decreasing functions $f(x)$ and $g(x)$ and defined on some domain D . For any two points a, b in the domain, we have $f(a) > f(b)$ and $g(a) > g(b)$. Now define the sum of these two functions $h(x) = f(x) + g(x)$. We have $h(a) = f(a) + g(a) > f(b) + g(b) = h(b)$ and so we have $h(a) > h(b)$ which means $h(x)$ must be a decreasing function as well.

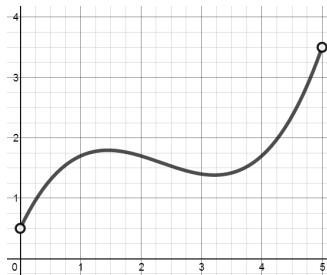
Solution: Almost-solution: Consider two differentiable decreasing functions $f(x)$ and $g(x)$ and defined on some domain D . We know that $f'(x) < 0$ and $g'(x) < 0$. Now consider the sum of the two functions $h(x) = f(x) + g(x)$. We know that $h'(x) = f'(x) + g'(x)$ and therefore $h'(x) < 0$. This means $h(x)$ is decreasing. **Important Note:** This is an almost solution as the question does not say that the functions have to be differentiable. We can have decreasing functions that are not differentiable. Tough question indeed.

7. For each of the following parts, sketch a continuous function with domain $(0, 5)$ satisfying the conditions:

2 marks

- (a) A function $f(x)$ with a local maximum, a local minimum but no global maximum and no global minimum.

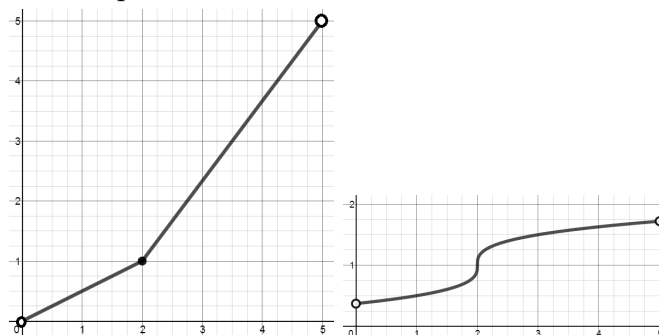
Solution: Some Examples:



2 marks

- (b) A function $g(x)$ with $g'(x) > 0$ for all $x \neq 2$ and $g'(2)$ undefined.

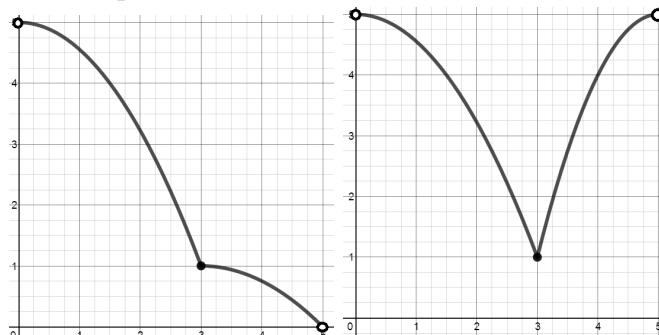
Solution: Some Examples:



3 marks

- (c) A function $h(x)$ with $h''(x) < 0$ for all $x \neq 3$. Further, $h'(3)$ does not exist and $h'(x) < 0$ for $x < 3$.

Solution: Some Examples:



This page may be used for rough work. It will not be marked.
