Homework 3, MATH 110-001

Due date: Friday, Feb 9, 2018 (in class)

Hand in full solutions to the questions below. Make sure you justify all your work and include complete arguments and explanations. Your answers must be clear and neatly written, as well as legible (no tiny drawings or micro-handwriting please!). Your answers must be stapled, with your name and student number at the top of each page.

1. Suppose two sprinters racing each other finish in a tie. Explain, using the Mean Value Theorem, why this means there must have been a moment in the race when the two sprinters were running at exactly the same speed.

2. Find the local extrema of the following functions by using the first derivative test. For part (a) also determine the inflection point and the concavity of the function. (You do not need to calculate the y-coordinates for local extrema and the inflection points.)

   (a) $f(x) = \frac{x}{2} - \arctan(x)$

   Hint: Note that $\arctan(x)$ is the inverse function for $\tan(x)$ i.e. $\arctan(\tan(x)) = x$, and its derivative is given by $(\arctan(x))' = \frac{1}{1+x^2}$.

   (b) $g(x) = -\frac{1}{5}(x-4)^{\frac{5}{3}} - 2(x-4)^{\frac{2}{3}}$

3. Find the critical numbers of the implicit function defined by the equation

$$x^2 + y^2 - 3xy + 5 = 0.$$

   You do not need to find the local extrema.

4. Sketch a graph for the function f that satisfies all the following conditions.

   (i) Domain: $[0, 6]$

   (ii) x-intercept $= 6$, y-intercept $= 4$

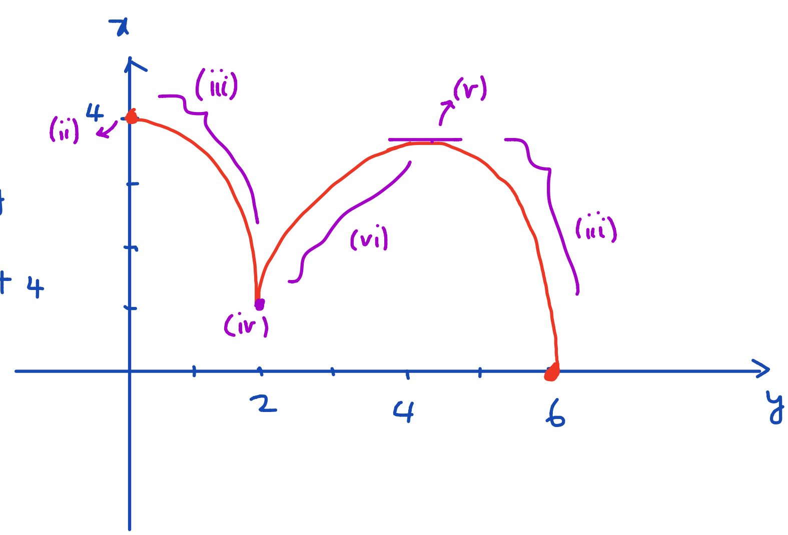
   (iii) $\underbrace{f'(x) < 0 \text{ when } x < 2 \text{ and } x > 4}$ decreasing curve

   (iv) $\underbrace{f'(2) \text{ does not exist.}}$ a cusp at 2

   (v) $\underbrace{f'(4) = 0}$ Horizontal tangent line at 4

   (vi) $\underbrace{f'(x) > 0 \text{ on } [2, 4]}$ increasing

   (vii) ~~$f''(x) > 0$ for all x in the domain~~



⊛ Note that we can't have a hole at $x=2$ because then $x=2$ is NOT part of domain which contradicts with Domain : $[0,6]$

There's a cusp / corner at $x=2$.

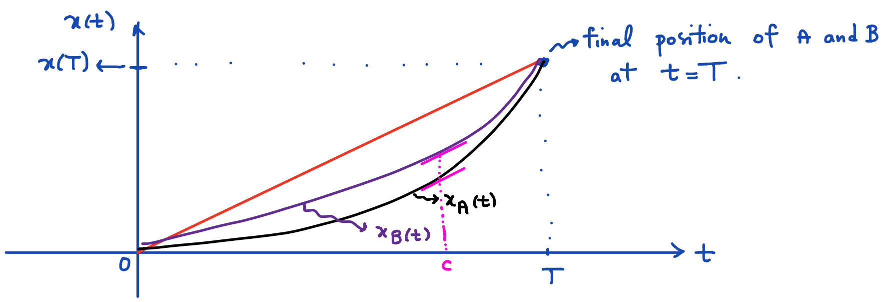
1. We first consider the position function for the two runners to be respectively $x_A(t)$ and $x_B(t)$, where t is the time since the start of the race. According to the question, both runners start and end the race at the same time. Let's take T to be the time at which they finish the race.

Let's verify conditions required for MVT:

1) The two runners run the entire race and there's no moment in time where they would "jump" any distance, so $x_A(t)$ and $x_B(t)$ are continuous functions.

2) The race trail doesn't come with a cusp and it's a smooth trail so the position function it's also differentiable.

To see better, let's draw a graph of the position function of the runners A and B.



Now we can apply MVT on the interval $[0,T]$:

⊛ Note that the tangent line to the graph of position at any time represents the velocity at that time.

Consider the difference function of positions: $D(t) = x_A(t) - x_B(t)$

$x_A(t)$ and $x_B(t)$ are continuous & differentiable so $D(t)$ is also continuous and diff'iable. Moreover; $D(0) = x_A(0) - x_B(0) = 0$

$$D(T) = x_A(T) - x_B(T) = 0$$

So by Rolle's Theorem there is a c in $(0,T)$ such that

$$D'(c) = 0 \Rightarrow x_A'(c) - x_B'(c) = 0$$

$$\Rightarrow V_A(c) = V_B(c) \Rightarrow \text{equal speed}$$

a) $f(x) = \frac{x}{2} - \arctan x$

Domain : All real numbers : $\mathbb{R}$ ($\arctan x$ is defined everywhere.)

Critical numbers :

$$f'(x) = \frac{1}{2} - \frac{1}{1+x^2}$$

$f'(x) = 0$

$$\frac{1}{2} - \frac{1}{1+x^2} = 0$$

Common denominator

$$\frac{1+x^2-2}{2(1+x^2)} = 0$$

top = 0

$$x^2 - 1 = 0$$

$$\Rightarrow (x-1)(x+1) = 0 \quad \Rightarrow \boxed{x=1} \;,\; \boxed{x=-1}$$

$f'(x)$ is NOT defined when

$1 + x^2 = 0$

$\Rightarrow x^2 = -1$ NEVER happens

Sign chart for f':

x	$-\infty$		-1		1		∞
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\nearrow$	local max at $x=-1$	$\searrow$	local min at $x=1$	$\nearrow$	

test $x=-2$ test $x=0$ test $x=2$

$$f'(x) = \frac{x^2-1}{1+x^2}$$

always $\oplus$

$$\longrightarrow \quad f'(2) = \frac{4-1}{+} > 0$$

$$f'(0) = \frac{0-1}{+} < 0$$

$$f'(-2) = \frac{4-1}{+} > 0$$

Summary from f':
- f is increasing in $(-\infty,-1)$ and $(1,+\infty)$
- f " decreasing " $(-1,1)$
- f has a local max at $x=-1$ and a local min at $x=1$.

Concavity: Find f'': Be strategic for your computation; choose easy rules and easy to differentiate forms.

$$f'(x) = \frac{1}{2} - \frac{1}{1+x^2} = \frac{x^2-1}{1+x^2}$$

which one is easier to differentiate? The first form: NO quotient rule needed

we can use power rule.

Rewrite in an easy form:

$$f'(x) = \frac{1}{2} - (1+x^2)^{-1}$$

$$f''(x) = -2x\cdot(-1)\cdot(1+x^2)^{-2} = 2x\,(1+x^2)^{-2} = \frac{2x}{(1+x^2)^2}$$

$$f''(x) = 0 \text{ when } 2x = 0 \Rightarrow x = 0$$

Sign chart for f'':

x	$-\infty$		0		∞
$f''(x)$		$-$	0	$+$	
$f(x)$		$\frown$	inflection point at $x=0$	$\smile$	

test $x=-1$ test $x=1$

$$f''(x) = \frac{2x}{(1+x^2)^2}$$

always $\oplus$

$$\longrightarrow \quad f''(1) = \frac{2}{+} > 0$$

$$f''(-1) = \frac{-2}{+} < 0$$

Summary from f'': f is concave down in $(-\infty,0)$ and concave up in $(0,\infty)$ and it has an inflection point at $x=0$.

b) $g(x) = -\frac{1}{5}(x-4)^{5/3} - 2(x-4)^{2/3}$

Domain: Write in form of roots: $g(x) = -\frac{1}{5}\sqrt[3]{(x-4)^5} - 2\sqrt[3]{(x-4)^2}$

It only involves odd roots which are defined everywhere $\Rightarrow$ Domain : $\mathbb{R}$

Critical numbers: Use the power form NOT the root form.

$$g'(x) = -\frac{1}{5}\cdot\frac{5}{3}(x-4)^{5/3-1} - 2\cdot\frac{2}{3}(x-4)^{2/3-1}$$

$$= -\frac{1}{3}(x-4)^{2/3} - \frac{4}{3}(x-4)^{-\frac{1}{3}}$$

$\to$ negative power must be written in positive forms.

$$= -\frac{1}{3}(x-4)^{2/3} - \frac{4}{3}\,\frac{1}{(x-4)^{1/3}}$$

$f'(x) = 0$: Common denominator

$x^a \cdot x^b = x^{a+b}$

$$g'(x) = \frac{-(x-4)^{2/3}\cdot(x-4)^{1/3} - 4}{3(x-4)^{1/3}}$$

$f'(x)$ is NOT defined when $\boxed{x=4}$.

This is a critical number. It's in the domain of the function but $g'(4)$ is not defined

$$= \frac{-(x-4)-4}{3(x-4)^{1/3}} = \frac{-x}{3(x-4)^{1/3}}$$

$\longrightarrow$ $g'(x)=0$ when $\boxed{x=0}$

Sign chart for g':

x	$-\infty$	test $x=-1$	0	test $x=1$	4	test $x=5$	∞
$g'(x)$		$-$	0	$+$	∞	$-$	
$g(x)$		$\searrow$	local min at $x=0$	$\nearrow$	local max at $x=4$	$\searrow$	

$$g'(x) = \frac{-x}{3\sqrt[3]{x-4}} \longrightarrow g'(5) = \frac{-5}{3\sqrt[3]{5-4}} = \frac{-}{+} < 0$$

$$g'(1) = \frac{-1}{3\sqrt[3]{1-4}} = \frac{-1}{3\sqrt[3]{-3}} = \frac{-}{-} > 0$$

$$g'(-1) = \frac{-(-1)}{3\sqrt[3]{(-1)-4}} = \frac{+}{-} < 0$$

Summary: f is increasing in $(0,4)$ & decreasing in $(-\infty,0)$ and $(4,\infty)$

3) $x^2+y^2-3xy+5=0$

implicit differentiation: $2x+2yy'-3(y+xy')=0$

$$2x-3y+y'(2y-3x)=0 \Rightarrow y'=\frac{3y-2x}{2y-3x}$$

Critical Numbers: $\begin{cases} y'=0 \xrightarrow{\text{top}=0} 3y-2x=0 \\ y' \text{ undefined} \xrightarrow{\text{bottom}=0} 2y-3x=0 \end{cases}$

We don't get a number, but in each case we get a relationship between x and y.

(1) $3y-2x=0$

$\Rightarrow y=\frac{2}{3}x$

Go to the original equation $\longrightarrow$ $x^2+\left(\frac{2}{3}x\right)^2-3x\left(\frac{2}{3}x\right)+5=0$

$\Rightarrow x^2+\frac{4}{9}x^2-2x^2+5=0$

$\Rightarrow \frac{9x^2+4x^2-18x^2}{9}=-5$

$\Rightarrow -\frac{5}{9}x^2=-5$

$\Rightarrow x^2=9$

$\Rightarrow \boxed{x=3}, \boxed{x=-3}$

(2) $2y-3x=0$

$y=\frac{3}{2}x$

$\longrightarrow x^2+\left(\frac{3}{2}x\right)^2-3x\left(\frac{3}{2}x\right)+5=0$

$\Rightarrow x^2+\frac{9}{4}x^2-\frac{9}{2}x^2+5=0$

$\Rightarrow \frac{4x^2+9x^2-18x^2}{4}=-5$

$\Rightarrow -\frac{5}{4}x^2=-5$

$\Rightarrow x^2=4$

$\Rightarrow \boxed{x=2}, \boxed{x=-2}$