

Lecture 17 .

Feb 9.

Be strategic in your computations.

In the following examples, we'd like to see what the useful strategies are in finding the 1st and 2nd derivative: f' and f'' .

We'd like to use the simplest differentiation rules and also an easy way to solve $f'(x) = 0$ and $f''(x) = 0$

Example 1 . $f(x) = \frac{6}{x^2 + 3}$

Domain: \mathbb{R} (because $x^2 + 3 = 0$ is when $x^2 = -3$ and it NEVER happens.)

Critical number & inflection points : We need to compute $f'(x)$ and $f''(x)$

$f'(x)$: easier to write $f(x)$ in a power form and NOT to apply the quotient rule.

$$f(x) = 6(x^2 + 3)^{-1} \Rightarrow f'(x) = -6 \cdot 2x \cdot (x^2 + 3)^{-2}$$

- Now solve $f'(x) = 0$ and $f'(x)$ NOT Defined.

Convert to positive exponent $f'(x) = \frac{-12x}{(x^2 + 3)^2}$

- To find $f''(x)$; it's easier to apply the quotient rule :

$$f''(x) = \frac{\underline{-12}(x^2 + 3)^2 - (\underline{-12x}) \cdot 2x \cdot 2\underline{(x^2 + 3)}}{(x^2 + 3)^4}$$

- Now solve $f''(x) = 0$ factoring comes first \rightarrow then distribute, expand ...

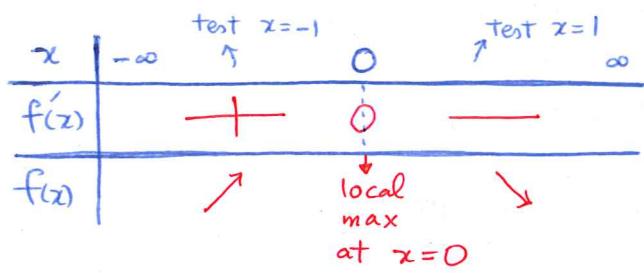
$$f''(x) = \frac{-12(x^2+3)(x^2+3 - 4x^2)}{(x^2+3)^4}$$

$$= \frac{-12(3 - 3x^2)}{(x^2+3)^3} = \frac{12 \cdot 3(x^2-1)}{(x^2+3)^3}$$

Exercise . Make the sign charts for f' and f'' and determine the intervals of increase , decrease and concavity in f .

$$f'(x) = \frac{-12x}{(x^2+3)^2} = 0 \Rightarrow x=0$$

$\rightarrow f'(x)$ is NOT defined when $x^2+3=0$ which is impossible.



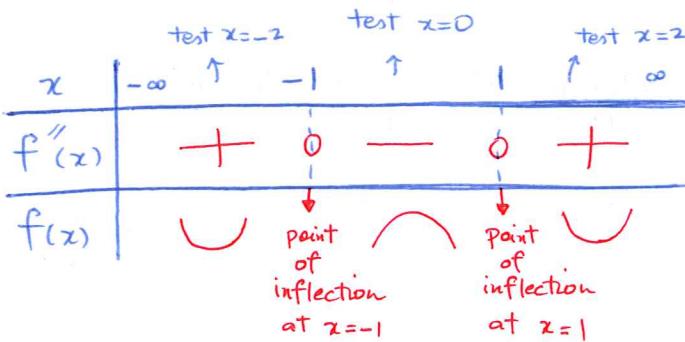
$$f'(x) = \frac{-12x}{(x^2+3)}$$

always \ominus
always \oplus

f is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$ and f has a local max at $x=0 \Rightarrow f(0) = \frac{6}{0+3} = 2$ \downarrow max value $\Rightarrow (0, 2)$ is the local maximum point.

$$f''(x) = \frac{36(x^2-1)}{(x^2+3)^3} = 0 \Rightarrow x^2-1 = (x-1)(x+1)=0 \Rightarrow x=1, x=-1$$

$\rightarrow f''(x)$ is Not defined when $x^2+3=0 \Rightarrow$ NOT possible



$$f''(x) = \frac{36(x-1)(x+1)}{(x^2+3)^3}$$

always \oplus

f is concave up in $(-\infty, -1)$ and $(1, \infty)$ and concave down in $(-1, 1)$.

$x=-1 \Rightarrow f(-1) = \frac{6}{1+3} = 6/4 = 3/2 = f(1) \Rightarrow (-1, 3/2), (1, 3/2)$ are inflection points.

Example 2 . $f(x) = x + \frac{9}{x}$

Domain . $x \neq 0$

Critical numbers : ∞
Inflection points :

(1) $f'(x) = 1 - 9x^{-2}$ positive exponent

for $f'(x)=0$ and $f'(x)$ NOT defined

(2) $f'(x) = 1 - \frac{9}{x^2}$ Needs more simplification : common denominator

(3) $f'(x) = \frac{x^2 - 9}{x^2}$

Now, we have 3 forms for f' , we should find the one which is easiest to differentiate to find f'' :

(1) is the easiest: $f''(x) = 18x^{-3}$ For $f''(x)=0 \Rightarrow f''(x) = \frac{18}{x^3}$

→ Sign chart for f' :

$$f'(x)=0 \xrightarrow[form]{3rd} x^2-9=(x-3)(x+3)=0 \Rightarrow \boxed{x=3}, \boxed{x=-3}$$

$f'(x)$ NOT defined when $x^2=0 \Rightarrow \boxed{x=0}$

x	$-\infty$	-4	-3	-2	0	2	3	4	∞
$f'(x)$	+	0	-	∞	-	0	+		
$f(x)$	↗	local max at $x=-3$	↘	↘ local min at $x=3$	↗				
$f'(x) = \frac{x^2 - 9}{x^2}$									

always \oplus $\Rightarrow f'(4) = \frac{4^2 - 9}{16} > 0, f'(-2) = \frac{4 - 9}{16} < 0$
 $f'(2) = \frac{2^2 - 9}{4} < 0, f'(-4) = \frac{16 - 9}{16} > 0 \Rightarrow$

sign chart for f'' :

$$f''(x)=0 \Rightarrow 18=0 \text{ impossible}$$

$f''(x)$ is NOT defined at $x=0$

x	$-\infty$	0	∞
$f''(x)$	-	∞	+
$f(x)$	↙	NOT an inflection point	↗

f is concave down in $(-\infty, 0)$ and concave up in $(0, \infty)$.

f is increasing in $(-\infty, -3)$ and $(3, \infty)$ and decreasing in $(-3, 0)$ and $(0, 3)$.

Some useful tips to minimize errors in computations.

→ In finding f' and f'' :

- Convert root functions and simple rational functions to powers first and then differentiate.

For example: $\sqrt[3]{x^2} = x^{2/3}$ or $\frac{1}{\sqrt{x+1}} = (x+1)^{-\frac{1}{2}}$

or $\frac{6}{(x^2+3)} = 6(x^2+3)^{-1}$

- going from f' to f'' : first check which form of f' is easier to differentiate. Sometimes, we need to use the expression for f' from a few steps back.

In example 2, f' has three forms, we used each form in different steps of computation.

→ In solving $f'(x)=0$ or $f''(x)=0$:

- First factorize, then distribute multiplication. However,

For example: factorized forms are usually harder to differentiate.

easy for differentiation $\leftarrow \frac{x^2-9}{x^2}$ vs. $\frac{(x-3)(x+3)}{x^2} \rightarrow$ easy to solve for $=0$

- Convert negative exponents to positive by writing them as the denominator in a fraction.

For example:

$$f'(x) = \begin{cases} 1 - 9x^{-2} & \rightarrow \text{Good for differentiation} \\ & \text{But} \\ 1 - \frac{9}{x^2} & \rightarrow \text{Good for the points at which the function is undefined.} \\ \frac{x^2-9}{x^2} & \rightarrow \text{Good for solving for } f'(x)=0 \end{cases}$$

→ In testing points :

- Choose the points that are easy to calculate. For special functions, choose the numbers for which the function value is known. $\ln e = 1$, $\ln 1 = 0$, $e^0 = 1$, ...
- Look for the terms that are always positive or negative, you don't need to check their sign.

For example :

$$\frac{x^2(x-1)}{x^2+3}$$

always \oplus

$\frac{-x^2-1}{x^3}$

\rightarrow just test for this , \rightarrow always 0
 \rightarrow just test this

* Note: Even exponents gives always-positive terms. Be careful for odd-exponents

→ In identifying critical numbers, local extrema, inflection point :

- The x -values that are not in the domain, they can NOT be critical numbers, extrema and inflection points either.
- BUT; they do come in our sign charts.

Compare:

$$f(x) = x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}} \quad \text{vs.} \quad f(x) = x^{\frac{2}{3}}$$

Domain = $x \neq 0$

Domain = \mathbb{R}

↓

$$f'(x) = -\frac{2}{3} x^{-\frac{5}{3}}$$

$$= -\frac{2}{3} \cdot \frac{1}{x^{\frac{5}{3}}}$$

↓
 $f'(x)$ ND at $x=0$ NOT a critical number.
(NOT in domain)

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$$

$$= \frac{2}{3} \cdot \frac{1}{x^{\frac{1}{3}}}$$

↓
 $f'(x)$ ND at $x=0$ Critical number
it is in domain