Lecture 17. Feb 9.

Be strategic in your computations.

In the following examples, we'd like to see what the useful strategies are in finding the 1^{st} and 2^{nd} derivative: f' and f''.

We'd like to use the simplest differentiation rules and also an easy way to solve $f'(x)=0$ and $f''(x)=0$

Example 1. $f(x) = \dfrac{6}{x^2+3}$

Domain: $\mathbb{R}$ (because $x^2+3=0$ is when $x^2=-3$ and it NEVER happens.)

Critical number & inflection points: we need to compute $f'(x)$ and $f''(x)$.

$f'(x)$: easier to write $f(x)$ in a power form and NOT to apply the quotient rule.

$$f(x) = 6(x^2+3)^{-1} \Rightarrow f'(x) = -6 \cdot 2x \cdot (x^2+3)^{-2}$$

- Now solve $f'(x)=0$ and $f'(x)$ NOT Defined.

$$\xrightarrow[\text{positive exponent}]{\text{Convert to}} \quad f'(x) = \frac{-12x}{(x^2+3)^2}$$

- To find $f''(x)$; it's easier to apply the quotient rule:

$$f''(x) = \frac{\boxed{-12}\,\underline{(x^2+3)}^2 - (\boxed{-12}x)\cdot 2x \cdot 2\underline{(x^2+3)}}{(x^2+3)^4}$$

- Now solve $f''(x)=0$ $\xrightarrow[\text{then distribute, expand} \cdots]{\text{factoring comes first}}$

$$f''(x) = \frac{-12(x^2+3)\left(x^2+3-4x^2\right)}{(x^2+3)^4}$$

$$= \frac{-12\left(3-3x^2\right)}{(x^2+3)^3} = \frac{12\cdot 3\,(x^2-1)}{(x^2+3)^3}$$

Exercise. Make the sign charts for f' and f'' and determine the intervals of increase, decrease and concavity in f.

$$f'(x) = \frac{-12x}{(x^2+3)^2} = 0 \Rightarrow x = 0$$

$\longrightarrow$ $f'(x)$ is NOT defined when $x^2+3=0$ which is impossible.

x	$-\infty$	test $x=-1$	0	test $x=1$	∞
$f'(x)$		$+$	0	$-$	
$f(x)$		$\nearrow$	local max at $x=0$	$\searrow$	

$$f'(x) = \frac{\overset{\text{always } \ominus}{(-12)}\,x}{\underset{\text{always } \oplus}{(x^2+3)}}$$

f is increasing in $(-\infty, 0)$ and decreasing in $(0,\infty)$ and f has a local max at $x=0 \Rightarrow f(0) = \frac{6}{0+3} = \underset{\text{max value}}{2} \Rightarrow (0,2)$ is the local maximum point.

$$f''(x) = \frac{36(x^2-1)}{(x^2+3)^3} = 0 \Rightarrow x^2-1 = (x-1)(x+1) = 0 \Rightarrow \boxed{x=1},\ \boxed{x=-1}$$

$\longrightarrow$ $f''(x)$ is Not defined when $x^2+3=0 \Rightarrow$ NOT possible

x	$-\infty$	test $x=-2$	-1	test $x=0$	1	test $x=2$	∞
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\cup$	point of inflection at $x=-1$	$\cap$	point of inflection at $x=1$	$\cup$	

$$f''(x) = \frac{\overset{\oplus}{(36)}(x-1)(x+1)}{\underset{\text{always } \oplus}{(x^2+3)^3}}$$

f is concave up in $(-\infty,-1)$ and $(1,\infty)$ and concave down in $(-1,1)$.

$x=-1 \Rightarrow f(-1) = \frac{6}{1+3} = 6/4 = 3/2 = f(1) \Rightarrow (-1, 3/2), (1, 3/2)$ are inflection points.

Example 2. $f(x) = x + \frac{9}{x}$

Domain. $x \neq 0$

Critical numbers & Inflection points:

Easy way to differentiate is to write $f(x)$ in the power form: $f(x) = x + 9x^{-1}$

(1) $f'(x) = 1 - 9x^{-2}$ positive exponent for $f'(x)=0$ and $f'(x)$ NOT defined

(2) $f'(x) = 1 - \frac{9}{x^2}$ Needs more simplification: common denominator

(3) $f'(x) = \frac{x^2-9}{x^2}$

Now, we have 3 forms for f', we should find the one which is easiest to differentiate to find f'':

(1) is the easiest: $f''(x) = 18x^{-3}$ $\xrightarrow{\text{For } f''(x)=0}$ $f''(x) = \frac{18}{x^3}$

Sign chart for f':

$f'(x)=0 \xrightarrow[\text{form}]{\text{3rd}} x^2-9=(x-3)(x+3)=0$

$\Rightarrow \boxed{x=3}, \boxed{x=-3}$

$f'(x)$ NOT defined when $x^2=0 \Rightarrow \boxed{x=0}$

x	$-\infty$	$x=-4$	-3	$x=-2$	0	$x=2$	3	$x=4$	∞
$f'(x)$		$+$	0	$-$	∞	$-$	0	$+$	
$f(x)$		$\nearrow$	local max at $x=-3$	$\searrow$		$\searrow$	local min at $x=3$	$\nearrow$	

$f'(x) = \frac{x^2-9}{x^2}$ always $\oplus$

$\Rightarrow f'(4) = \frac{4^2-9}{16} > 0$, $f'(-2) = \frac{4-9}{+} < 0$

$f'(2) = \frac{2^2-9}{+} < 0$, $f'(-4) = \frac{16-9}{16} > 0$

$\Rightarrow$ f is increasing in $(-\infty,-3)$ and $(3,\infty)$ and decreasing in $(-3,0)$ and $(0,3)$.

Sign chart for f'':

$f''(x)=0 \Rightarrow 18=0$ impossible

$f''(x)$ is NOT defined at $x=0$

x	$-\infty$		0		∞
$f''(x)$		$-$	∞	$+$	
$f(x)$		$\frown$	NOT an inflection point	$\smile$	

f is concave down in $(-\infty,0)$ and concave up in $(0,\infty)$.

Some useful tips to minimize errors in computations.

→ In finding f' and f'':

- Convert root functions and simple rational functions to powers first and then differentiate.

For example: $\sqrt[3]{x^2} = x^{2/3}$ or $\frac{1}{\sqrt{x+1}} = (x+1)^{-\frac{1}{2}}$

or $\frac{6}{(x^2+3)} = 6(x^2+3)^{-1}$

- going from f' to f''; first check which form of f' is easier to differentiate. Sometimes, we need to use the expression for f' from a few steps back.

In example 2, f' has three forms, we used each form in different steps of computation.

→ In solving $f'(x) = 0$ or $f''(x) = 0$:

- First factorize, then distribute multiplication. However, factorized forms are usually harder to differentiate.

For example:

easy for differentiation $\leftarrow \frac{x^2-9}{x^2}$ vs. $\frac{(x-3)(x+3)}{x^2} \rightarrow$ easy to solve for $=0$

- Convert negative exponents to positive by writing them as the denominator in a fraction.

For example:

$$f'(x) = \begin{cases} 1 - 9x^{-2} & \rightsquigarrow \text{Good for differentiation} \\ & \quad \text{But} \\ 1 - \frac{9}{x^2} & \rightsquigarrow \text{Good for the points at which the function is undefined.} \\ \frac{x^2-9}{x^2} & \rightsquigarrow \text{Good for solving for } f'(x) = 0 \end{cases}$$

→ In testing points :

- Choose the points that are easy to calculate. For special functions, choose the numbers for which the function value is known. $\ln e = 1$, $\ln 1 = 0$, $e^0 = 1$, ...

- Look for the terms that are always positive or negative, you don't need to check their sign.

For example : $\dfrac{x^2(x-1)}{x^2+3}$ → just test for this. always ⊕ , $\dfrac{-x^2-1}{x^3}$ → always ⊖ → just test this

⊛ Note : Even exponents gives always-positive terms. Be careful for odd-exponen

→ In identifying critical numbers, local extrema, inflection point :

- The x-values that are not in the domain, they can NOT be critical numbers, extrema and inflection points either.

  BUT; they do come in our sign charts.

Compare :

$f(x) = x^{-2/3} = \dfrac{1}{x^{2/3}}$ vs. $f(x) = x^{2/3}$

Domain = $x \neq 0$ ⇓ Domain $= \mathbb{R}$

$f'(x) = -2/3\, x^{-5/3}$ $f'(x) = \dfrac{2}{3} x^{-1/3}$

$= -\dfrac{2}{3} \dfrac{1}{x^{5/3}}$ $= \dfrac{2}{3} \dfrac{1}{x^{1/3}}$

⇓ ⇓

$f'(x)$ ND at $\boxed{x=0}$ NOT a critical number. (NOT in domain) $f'(x)$ ND at $\boxed{x=0}$ Critical number, it's in domain.