Lecture 18
Feb 14

Asymptotic Behavior of a Function

⇝ Vertical Asymptote

In the last few lectures, we learned how to get info about the shape of f by checking the sign of f' and f''.

We saw that:

Review:

f'	$\oplus$	$\Leftrightarrow$	f increasing
f'	$\ominus$	$\Leftrightarrow$	f decreasing

and local extrema is when f' changes sign.

$$\left.\begin{array}{ccc} f' & + & - \\ f & \nearrow & \searrow \end{array}\right\}\ \text{local max} \qquad \left.\begin{array}{ccc} f' & - & + \\ f & \searrow & \nearrow \end{array}\right\}\ \text{local min}$$

f''	$\oplus$	$\Leftrightarrow$	f concave up
f''	$\ominus$	$\Leftrightarrow$	f concave down

and inflection point is when f'' changes sign.

⟶ Note that in all these computations, we should always make sure that the points that we've found as local extrema or inflection points are included in the domain of the original function.

⟶ In this lecture, we learn how to get info from the function f itself by checking the behaviour of the function at infinity

Recall from Term 1 the following limits:

Example 1. $\lim_{x \to 3} \frac{-x+1}{x^2-9}$

As we know the first step in finding a limit is substitution which gives: $\lim_{x \to 3} \frac{-x+1}{x^2-9} = \frac{-3+1}{3^2-9} = \frac{-2}{0} = \pm\infty$

Now we need to split the limit into one-sided limits: $x \to 3^+$ and $x \to 3^-$ to determine the sign of ∞. Also, it is better to factor x^2-9 to $(x-3)(x+3)$:

$$\lim_{\underbrace{x \to 3^+}_{x=3.1}} \frac{-x+1}{(x-3)(x+3)} = \frac{-3+1}{0^+ \times 6} = \frac{-2}{0^+} = -\infty$$

$(3.1-3)=0.1=0^+$

and

$$\lim_{\underbrace{x \to 3^-}_{x=2.9}} \frac{-x+1}{(x-3)(x+3)} = \frac{-3+1}{0^- \times 6} = \frac{-2}{0^-} = +\infty$$

$(2.9-3)=-0.1=0^-$

Similarly when $x \to -3$ we have

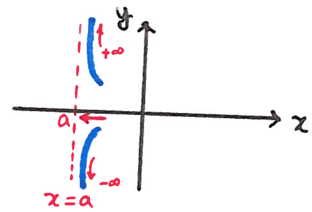
$$\lim_{x \to -3} \frac{-x+1}{(x-3)(x+3)} = \begin{cases} \lim_{\underbrace{x \to -3^+}_{x=-2.9}} \frac{-x+1}{(x-3)(x+3)} = \frac{-(-3)+1}{-6 \times 0^+} = \frac{4}{0^-} = -\infty \\ \lim_{\underbrace{x \to -3^-}_{x=-3.1}} \frac{-x+1}{(x-3)(x+3)} = \frac{4}{-6 \times 0^-} = \frac{4}{0^+} = +\infty \end{cases}$$

As you see in this example, x is approaching a fixed number while $f(x)$ approaches $+\infty$ or $-\infty$, this means that the function values become very very large or small. The fixed number to which x is approaching is called the vertical asymptote of f.

Definition. (Vertical Asymptote : VA)

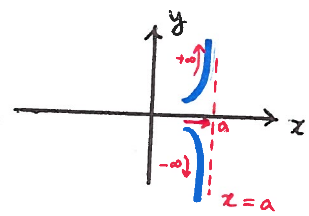
The vertical line $x=a$ is the vertical asymptote of the graph of the function $f(x)$ if one of the following one-sided limits happen

$$\boxed{\lim_{x \to a^+} f(x) = +\infty \quad \text{or} \quad -\infty}$$



Or

$$\boxed{\lim_{x \to a^-} f(x) = +\infty \quad \text{or} \quad -\infty}$$



By the definition, from example 1 :

If $f(x) = \dfrac{-x+1}{x^2-9}$ then $x=3$ and $x=-3$ are the VA of f.

Example 2. Find VA of $f(x) = \dfrac{x^2+5}{x(x-2)^2}$.

The candidates for VA's are the roots for $x(x-2)^2$ i.e. when $x(x-2)^2=0$

$\Rightarrow \quad x=0$ and $x=2$

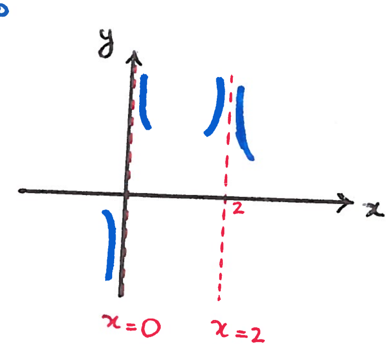
Now, let's check the limits for each x.

$$\lim_{x \to 0^+} \frac{x^2+5}{x(x-2)^2} = \frac{0+5}{0^+ \cdot (0-2)^2} = \frac{5}{0^+} = +\infty$$

Similarly $\lim_{x \to 0^-} f(x) = -\infty$

$$\lim_{x \to 2^+} \frac{x^2+5}{x(x-2)^2} = \frac{2^2+5}{2\underbrace{(2^+-2)^2}_{\oplus}} = \frac{9}{2 \times 0^+} = +\infty$$

$$\lim_{x \to 2^-} \frac{x^2+5}{x(x-2)^2} = \frac{9}{2\underbrace{(2^--2)^2}_{\oplus}} = \frac{9}{0^+} = +\infty$$



graph of f near the VA.

Remark. For a rational function $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, the candidates for vertical asymptotes are the roots of the denominator where $q(x)=0$.

Example 3. Find VA of $f(x) = \frac{x-1}{x^2+2x-3}$.

Candidates for the VA are when $x^2+2x-3=0$

$$\xrightarrow{\text{factor}} (x+3)(x-1)=0$$

$$\Longrightarrow x=-3, \quad x=1$$

Now we check the limits:

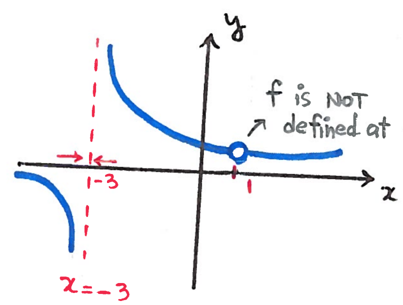
$$\lim_{x\to -3} \frac{x-1}{x^2+2x-3} = \lim_{x\to -3} \frac{\cancel{x-1}}{(\cancel{x-1})(x+3)} = \lim_{x\to -3} \frac{1}{x+3}$$

$$= \begin{cases} \lim_{x\to -3^+} \frac{1}{x+3} = \frac{1}{0^+} = +\infty \\ \lim_{x\to -3^-} \frac{1}{x+3} = \frac{1}{0^-} = -\infty \end{cases}$$

So $x=-3$ is a VA.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x-1}{(x-1)(x+3)} = \lim_{x\to 1} \frac{1}{x+3} = \frac{1}{4}$$

So $x=1$ is NOT a VA.



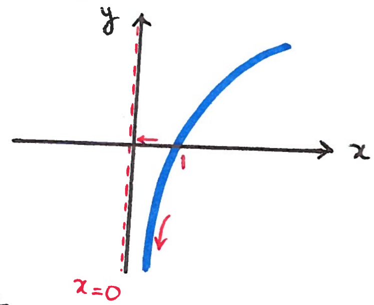
→ Conclusion: Roots of the dominator are only the possible VA, we need to check the limits to make sure if there's any cancellation.

In the next example, we find the VA of a different type of function.

Example 4. Find the VA of $f(x) = \ln x$.

Recall that the domain of $y = \ln x$ is $x > 0$, in fact $\ln x$ is NOT defined at $x = 0$, this suggests that $x = 0$ can be a VA of $\ln x$. Note that the graph of $\ln x$ looks like :

$$\lim_{x \to 0^+} \ln x = -\infty$$



the case $x \to 0^-$ NEVER happens, because the function never goes to the left of x. (NOT defined for negative values

Exercise. Find the VA of the following function.

(1) $f(x) = \ln(x-6)$.

Domain : $x - 6 > 0 \Rightarrow x > 6$ so $\lim_{x \to 6^+} \ln(x-6) = -\infty \Rightarrow x = 6$ VA

(2) $f(x) = \ln(2-x)$

Domain : $2 - x > 0 \Rightarrow x < 2$ so $\lim_{x \to 2^-} \ln(2-x) = -\infty \Rightarrow x = 2$ VA

(3) $f(x) = e^{\left(\frac{1}{x-3}\right)}$

Domain : $x \neq 3$. This a tricky example, as we should be careful in finding the limit when $x \to 3^+$ and 3^-.

$$\lim_{x \to 3^+} e^{\frac{1}{x-3}} = \lim_{x \to 3^+} e^{\frac{1}{3^+-3}} = \lim_{x \to 3^+} e^{\frac{1}{0^+}} = e^{+\infty} = +\infty \Rightarrow x = 3 \text{ is a VA}$$

$$\lim_{x \to 3^-} e^{\frac{1}{x-3}} = \lim_{x \to 3^-} e^{\frac{1}{3^- - 3}} = \lim_{x \to 3^-} e^{0^-} = e^{-\infty} = 0$$ $\hookrightarrow$ One-sided VA.