Lecture 19

Feb 16

# Asymptotic Behaviour of the function $\rightsquigarrow$ Horizontal Asymptote

Last lecture, we learned that a vertical asymptote is the line $x = a$ around which $f(x)$ becomes very large or small:

$x \to a$ $(x \to a^+$ or $x \to a^-)$ then $f(x) \to +\infty$ or $-\infty$

This lecture we see how a function behaves when $x \to +\infty$ or $x \to -\infty$.

**Example 1.** $\lim\limits_{x \to +\infty} x+1 =$

We don't know what $\infty$ is exactly but we know when $x \to +\infty$, it means that $x$ is becoming very large also $x+1$ becomes very large so as $x \to \infty$ then $x+1 \to \infty$ as well, so

$$\lim_{x \to \infty} x+1 = \infty$$

**Example 2.** $\lim\limits_{x \to +\infty} \frac{1}{x}$

$x \to +\infty$ means $x$ is getting larger and larger so we are dividing 1 by a large growing number:

$$\underbrace{\frac{1}{100} \cdots \frac{1}{1000} \cdots \frac{1}{1000000} \cdots \frac{1}{100\cdots000}}_{\text{approaches } 0}$$

$\rightsquigarrow$ the denominator keeps growing so the fraction becomes smaller and smaller

So $\lim_{x \to +\infty} \frac{1}{x} = 0$

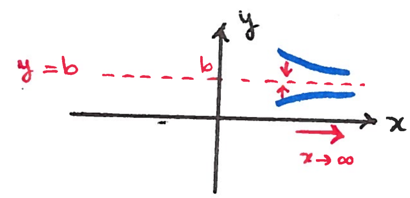
Similarly $\lim_{x \to -\infty} \frac{1}{x} = 0$ (since $-\frac{1}{1000000}, \dots, -\frac{1}{100\dots000}$) approaches 0 from negative values.

These limit leads us to another type of asymptote:

**Definition.** (Horizontal Asymptote : HA)

The horizontal line $y=b$ is the horizontal asymptote of the graph of the function $f(x)$ if one of the following occurs:

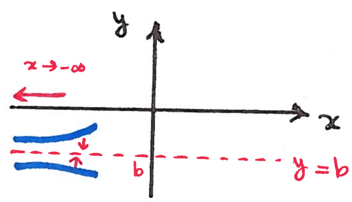
$$\boxed{\lim_{x \to +\infty} f(x) = b} \rightarrow$$




as $x \to +\infty$, one of these cases happen $f(x)$ approaches to $b$.

or

$$\boxed{\lim_{x \to -\infty} f(x) = b} \rightarrow$$

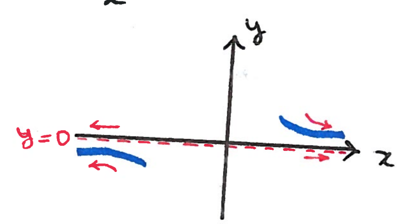



$x$ approaches $-\infty$ $f(x)$ approach $b$.

**Example 3.** Find the horizontal asymptote of $y = \frac{1}{x}$.

$\lim_{x \to +\infty} \frac{1}{x} = 0 \Rightarrow y = 0$ is HA.

also $\lim_{x \to -\infty} \frac{1}{x} = 0$




Similarly: $\lim_{x \to \pm\infty} \frac{1}{x^2} = 0$, $\lim_{x \to \pm\infty} \frac{2}{x^3} = 0$, $\lim_{x \to \pm\infty} \frac{-5}{x^8} = 0$

$$\lim_{x \to \pm\infty} \frac{1}{\sqrt[5]{x}} = 0$$

Because in all of these cases, a fixed number is divided by a growing large number.

**Remark**. Consider the function $f(x) = \dfrac{c}{x^r}$ where $c$ is a constant number and $r$ is a positive number, then

$$\lim_{x \to \pm\infty} \frac{c}{x^r} = 0$$ , this means that $y = 0$ is a HA of $f(x)$.

**Question**. How to find the limit of power functions in general when $x \to +\infty$ or $x \to -\infty$ ?

Example 1.

$$\lim_{x \to +\infty} x^3 + 2x^2 + 1$$ ⟶ compare the magnitude of each term :

As $x$ gets larger and larger which of the three term $x^3$, $2x^2$ and $1$ are the largest ? $x^3$

⟶ $x^3$ is the dominant term.

⟶ Factor out the dominante term.

$$\lim_{x \to \infty} \underbrace{x^3}_{\text{dominant}} + 2x^2 + 1 = \lim_{x \to \infty} x^3 \left( \frac{x^3}{x^3} + \frac{2x^2}{x^3} + \frac{1}{x^3} \right)$$

factoring means dividing

$$= \lim_{x \to \infty} x^3 \left( 1 + \underbrace{\cancel{\frac{2}{x}}}_{0} + \underbrace{\cancel{\frac{1}{x^3}}}_{0} \right)$$

* $\lim_{x \to \infty} \frac{2}{x} = 0$
* $\lim_{x \to \infty} \frac{1}{x^3} = 0$

$$= \lim_{x \to +\infty} x^3 \cdot 1$$

$$= \lim_{x \to +\infty} x^3 = +\infty$$

→ What is the horizontal asymptote of $f(x) = x^3 + 2x^2 + 1$ ?

Since $\lim\limits_{x \to +\infty} x^3 + 2x^2 + 1 = +\infty$ so this function has NO HA.

Similar method applies to $\lim\limits_{x \to -\infty} x^3 + 2x^2 + 1 = \lim\limits_{x \to -\infty} x^3 = -\infty$

**Example 2**. Find HA of $f(x) = \dfrac{4x^2 + 2x + 3}{-x^2 + 1}$

We need to find $\lim\limits_{x \to \pm\infty} f(x)$.

dominant

$$\lim_{x \to \infty} \frac{4x^2 + 2x + 3}{-x^2 + 1} = \lim_{x \to \infty} \frac{\cancel{4x^2}\left(\frac{4x^2}{4x^2} + \frac{2x}{4x^2} + \frac{3}{4x^2}\right)}{-\cancel{x^2}\left(\frac{-x^2}{-x^2} + \frac{1}{-x^2}\right)}$$

dominant

$$= \lim_{x \to \infty} \frac{4\left(1 + \cancel{\frac{1}{2x}}^{\,0} + \cancel{\frac{3}{4x^2}}^{\,0}\right)}{-1\left(1 - \cancel{\frac{1}{x^2}}_{\,0}\right)} = \frac{4}{-1} = -4$$

So $y = -4$ is a HA of $f$.

Similarly

$$\lim_{x \to -\infty} \frac{4x^2 + 2x + 3}{-x^2 + 1} = \lim_{x \to -\infty} \frac{\cancel{4x^2}\left(1 + \cancel{\frac{1}{2x}}_{\,0} + \cancel{\frac{3}{4x^2}}_{\,0}\right)}{-\cancel{x^2}\left(1 - \cancel{\frac{1}{x^2}}_{\,0}\right)} = -4$$

**Example 3**. Find HA of $f(x) = \dfrac{3x + x^3 - 5}{\sqrt{x} - 3x^4}$

dominant

dominant

$$\lim_{x \to +\infty} \frac{\cancel{x^3}\left(\frac{3x}{x^3} + \frac{x^3}{x^3} - \frac{5}{x^3}\right)}{3x^{\cancel{4}}\left(\frac{\sqrt{x}}{3x^4} - \frac{3x^4}{3x^4}\right)} = \lim_{x \to +\infty} \frac{\left(\cancel{\frac{3}{x^2}}_{\,0} + 1 - \cancel{\frac{5}{x^3}}^{\,0}\right)}{3x\left(\cancel{\frac{1}{3x^{7/2}}}_{\,0} - 1\right)} = \lim_{x \to +\infty} \frac{1}{-3x} = 0$$

$$* \quad \frac{x^{1/2}}{3x^4} = \frac{1}{3x^{4 - 1/2}} = \frac{1}{3x^{7/2}}$$

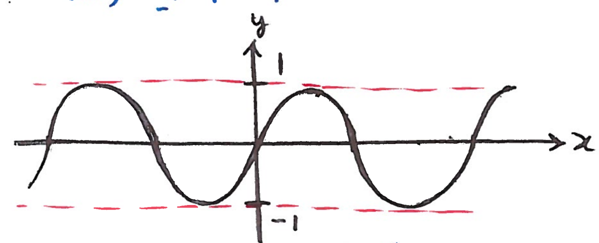
$\Rightarrow y = 0$ is HA.

Exercise. Find $\lim_{x \to -\infty} f(x)$ in Example 3 ?

Example 4. Find HA of $y = \frac{\sin(x)}{x}$.

$\sin(x)$ is NOT a power function and the factoring method is NOT working in this example. However, $\sin(x)$ is a special function with important properties. One important feature of $\sin(x)$ is that for any angle $x$; $\sin(x)$ is always between $-1$ and $1$ i.e. $-1 \leq \sin(x) \leq 1$.

Recall the graph of $\sin(x)$.




The function oscillates between $(-1)$ and $1$.

So

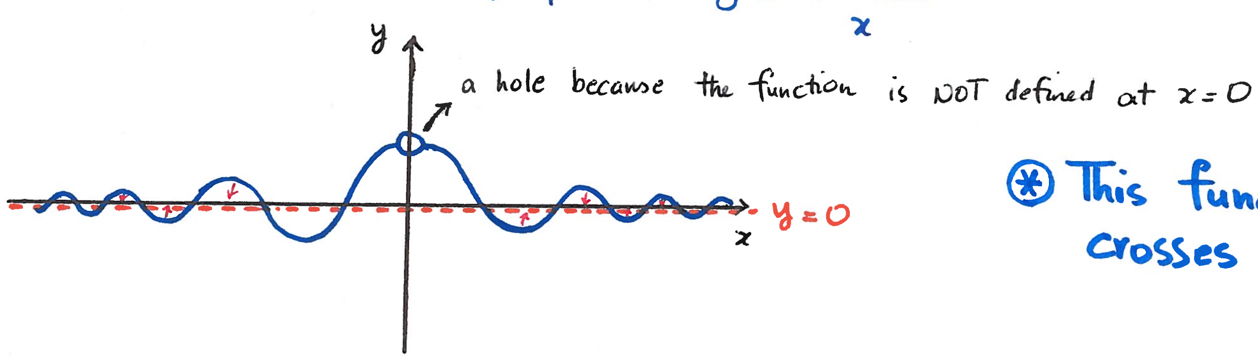
$$\lim_{x \to +\infty} \frac{\sin(x)}{x} = \frac{\text{a number between } -1 \text{ and } 1}{x}$$

the denominator grows, but numerator stays bounded between $(-1)$ and $1$.

$$= 0$$

Similarly; $\lim_{x \to -\infty} \frac{\sin(x)}{x} = 0 \implies y = 0$ is HA of the function $\frac{\sin(x)}{x}$.

Let's look at the graph of $y = \frac{\sin(x)}{x}$




⊛ This function crosses its HA.

**Conclusion** : A function CAN cross its horizontal asymptote.
( last example. )

**Question** . Is it possible that a function crosses its VA ?

→ Another Special function : $y = e^x$ .

**Example 5** . Find HA of $f(x) = e^x$ .

First, check the $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^x$

Let's see how $e^x$ behaves as $x$ grows larger and larger.

Recall that $e \approx 2.718$ so

$x = 1000 \rightsquigarrow e^{1000} \approx (2.7)^{1000}$

$x = 1000000 \rightsquigarrow e^{1000,000} \approx (2.7)^{1000,000}$

$\vdots$

$x = 100\ldots\infty \rightsquigarrow e^{1,000,\ldots,000} \approx (2.7)^{1,000,\ldots,000}$

$x \downarrow \infty$ ; larger & larger

As $x$ grows, $e^x$ grows too so $\boxed{\lim_{x \to \infty} e^x = \infty}$ .

What about $\lim_{x \to -\infty} e^x$

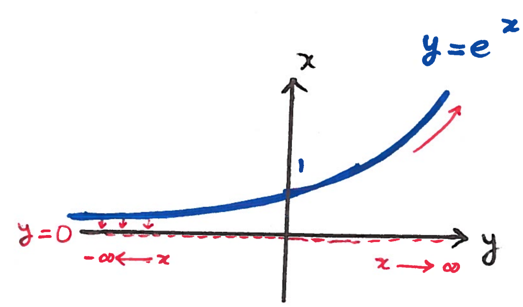
$x = -100 \rightsquigarrow e^{-100} = \dfrac{1}{e^{100}}$

$x = -10000 \rightsquigarrow e^{-10000} = \dfrac{1}{e^{10000}}$

$\vdots$

$x = -1000\ldots000 \rightsquigarrow e^{-1000\ldots000} = \dfrac{1}{e^{1000\ldots000}}$

becomes smaller & smaller




* See that graph of $e^x$ grows as $x \to \infty$ but approaches to 0 as $x \to -\infty$ .

So $\boxed{\lim_{x \to -\infty} e^x = 0}$ ⇒ $y=0$ is HA at the left side of the x axis