Lecture 13 and 14 Jan 31, Feb 2

In last lecture, we learned about the important relationship between the function, f, and its derivative f'.

if f is increasing in an interval (a,b) $\Rightarrow$ then $f'(x)$ is positive for all x in (a,b). $(f'(x) \geq 0)$

if f is decreasing in (a,b) $\Rightarrow$ $f'(x)$ is negative for all x in (a,b) $(f'(x) \leq 0)$

And vice versa:

if $f'(x) > 0$ for all x in (a,b) $\Rightarrow$ $f(x)$ is increasing in (a,b).

if $f'(x) < 0$ for all x in (a,b) $\Rightarrow$ $f(x)$ is decreasing in (a,b).

We use this property to determine the intervals in which a function decreases and increases.
Then, by checking the sign of the derivative, we also find the local extrema of a function.

Examples. For the following functions, Find the intervals in which the function is increasing/decreasing and then find the local extrema.

Note: A very important point in such questions is to be organized and very very careful in your computation. Double (triple) check your algebra especially when you differentiate. Try to be as neat as possible in order to effectively keep track of the calculation at each step.

1) $f(x) = 2x^5 + 5x^4 - 10x^3$

Step 1 : Domain : Domain f = all real numbers = $\mathbb{R}$

(a polynomial is defined everywhere.)

Step 2 : Critical numbers : (x's such that $f'(x) = 0$ & $f'(x)$ is NOT defined.)

$f'(x) = 10x^4 + 20x^3 - 30x^2$

$f'(x) = 0$

$f'(x)$ NOT defined → NEVER happens. f' is defined everywhere.

$\Downarrow$

$10x^4 + 20x^3 - 30x^2 = 0$

$\Downarrow$ factor

$10x^2(x^2 + 2x - 3) = 0$

$\Downarrow$

$10x^2 = 0$ or $x^2 + 2x - 3 = 0$

$\Downarrow$ $\qquad$ $\Downarrow$

$x^2 = 0$ $\qquad$ $(x+3)(x-1) = 0$

$\Downarrow$ $\qquad$ $\Downarrow$

$\boxed{x = 0}$ $\qquad$ $\boxed{x = -3}$, $\boxed{x = 1}$

Step 3 : Use the critical numbers to make a sign chart for f'.

Test a point in each interval, plug it into f' and find the sign.

Critical numbers sit here in an increasing order

x	$-\infty$	test $x=-4$, $f'(-4)>0$	-3	test $x=-1$, $f'(-1)<0$	0	test $x=\frac{1}{2}$, $f'(\frac{1}{2})<0$	1	test $x=2$, $f'(2)>0$	$+\infty$
$f'(x)$		$+$	0	$-$	0	$-$	0	$+$	
$f(x)$		$\nearrow$	local Max	$\searrow$		$\searrow$	local Min	$\nearrow$	

Testing points (plug them into f'):

Write f' in the most factorized form so that your computation becomes easier.

$$f'(x) = 10\,x^2(x+3)(x-1)$$

$$f'(2) = \underset{+}{10\cdot(2)^2}\cdot\underset{+}{(2+3)}\underset{+}{(2-1)} > 0$$

$$f'\left(\tfrac{1}{2}\right) = \underset{+}{10\cdot\left(\tfrac{1}{2}\right)^2}\cdot\underset{+}{\left(\tfrac{1}{2}+3\right)}\underset{-}{\left(\tfrac{1}{2}-1\right)} < 0$$

$$f'(-1) = \underset{+}{10\,(-1)^2}\cdot\underset{+}{(-1+3)}\underset{-}{(-1-1)} < 0$$

$$f'(-4) = \underset{+}{10\cdot(-4)^2}\cdot\underset{-}{(-4+3)}\cdot\underset{-}{(-4-1)} > 0$$

Step 4: Use the sign for f' to determine whether f is increasing or decreasing.

$\rightsquigarrow$ Add another row to the sign chart above and denote increasing by $\nearrow$ and decreasing by $\searrow$.

Intervals
- f is increasing in $\boxed{(-\infty,-3)}$ and $\boxed{(1,+\infty)}$
- f is decreasing in $\boxed{(-3,0)}$ and $\boxed{(0,1)}$

Note . Never close a bracket for $\pm\infty$.

if the derivative is 0 at any of the critical numbers, it's OK if you close the bracket at that point. For this example, it's also correct to write : $(-\infty,-3]$ and $[1,+\infty)$ f is increasing and $[-3,0]$ and $[0,1]$ f is decreasing.

Step 5 : Determine the points at which f has a local min or max. To do this, we use the

First Derivative Test

If f' changes sign around a critical number, then there is a local extremum at that number.

If f' changes sign from positive $\oplus$ to negative $\ominus$

then $f(x)$ changes from increasing $\nearrow$ to decreasing $\searrow$

$\Longrightarrow$ There is a local Maximum

If f' changes sign from $\ominus$ to $\oplus$

then f changes from $\searrow$ to $\nearrow$

$\Longrightarrow$ There is a local Minimum.

For this example :

local min at $x=-3$ $\xrightarrow{\text{y-Coordinate}}$ $f(-3)=2(-3)^5+5(-3)^4-10(-3)^3$
$=189$ $\Rightarrow$ $\boxed{(-3,189)}$

local max at $x=1$ $\longrightarrow$ $f(1)=2+5-10=-3$ $\Rightarrow$ $\boxed{(1,-3)}$

2) $f(x) = \frac{2}{x} + x^2$

This should come in the sign chart too.

Domain : Everywhere except $x=0$ or $x \neq 0$

Critical Numbers : Try to be strategic in applying differentiation Rules, the less computations the better and less possibility of making algebra mistakes.

Write the first term with a negative power & apply power rule (much easier than quotient rule.)

$f(x) = 2x^{-1} + x^2$

$\Rightarrow f'(x) = 2.(-1)\, x^{-2} + 2x$

$f'(x) = \frac{-2}{x^2} + 2x$

always make negative exponents positive for finding critical numbers.

$f'(x) = 0$

Common denominator

$f'(x) = \frac{-2 + 2x^3}{x^2}$

$f'(x)=0$ when the numerator $=0$

$-2 + 2x^3 = 0$

$2x^3 = 2$

$x^3 = 1 \Rightarrow x = \sqrt[3]{1} = 1$

$f'(x)$ is NOT defined at $x=0$. But $x=0$ is NOT in the domain of the original function, so we don't take $x=0$ as a critical number, but we need to consider it in our sign chart. (In curve sketching, it's going to be important.)

Make a sign chart.

f' is NOT defined at $x=0$

f' is 0 at $x=1$

test $x=2$

x	$-\infty$	test $x=-1$	0	test $\frac{1}{2}$	1		$+\infty$
$f'(x)$		$-$	∞	$-$	0	$+$	
$f(x)$		$\searrow$		$\searrow$	local min	$\nearrow$	

$$f'(x) = \frac{-2}{x^2} + 2x$$

Test points

$$f'(2) = \frac{-2}{(2)^2} + 2\cdot 2 = \frac{-2}{4} + 4 = -\frac{1}{2} + 4 > 0$$

$$f'\left(\tfrac{1}{2}\right) = \frac{-2}{\left(\frac{1}{2}\right)^2} + 2\cdot\left(\tfrac{1}{2}\right) = \frac{-2}{\frac{1}{4}} + 1 = -8+1 = -7 < 0$$

$$f'(-1) = \frac{-2}{(-1)^2} + 2\cdot(-1) = \frac{-2}{1} - 2 = -4 < 0$$

Intervals: f is increasing in $(1, +\infty)$ (or $[1, \infty)$)

and f is decreasing in $(-\infty, 0)$ and $(0, 1)$

Since 0 is not in the domain we use open brackets at 0.

Local extrema: f has a local min at $x=1$

$\xrightarrow{\text{y-coordinate}}$ $f(1) = \frac{2}{1} + (1)^2 = 3 \Rightarrow (1,3)$ is the local min point.

3) $g(x) = 5\,x^2\,e^{(9-3x)}$

Domain: All real numbers: $\mathbb{R}$ ($g(x)$ is defined everywhere.)

Critical numbers:

$$g'(x) = 5\left(2x\,e^{9-3x} + \underbrace{x^2(-3)\cdot e^{9-3x}}_{\text{chain rule for } \left(e^{9-3x}\right)'}\right)$$

product rule

$$\xrightarrow[g']{\text{simplify}} \quad g'(x) = 5\left(2x e^{9-3x} - 3x^2 e^{9-3x}\right)$$

Factor $\leftarrow$ $= 5\,x\,e^{9-3x}\,(2 - 3x)$

$g'(x) = 0$ $\qquad\qquad$ $g'(x)$ is defined everywhere.

$\Downarrow$

$5x = 0 \longrightarrow \boxed{x = 0}$

or $\quad e^{9-3x} = 0 \longrightarrow$ NEVER happens. $e^{\text{▨}}$ always $\oplus$

or $\quad 2 - 3x = 0 \longrightarrow \boxed{x = \frac{2}{3}}$

Sign chart:

x	$-\infty$	test $x=-1$	0	test $x=\frac{1}{3}$	$\frac{2}{3}$	test $x=1$	$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$		$\searrow$	local min	$\nearrow$	local max	$\searrow$	

$$g'(1) = \underset{+}{5}\cdot \underset{+}{1}\cdot e^{9-3}\,\underset{-}{(2-3)} < 0$$

$$g'\left(\tfrac{1}{3}\right) = \underset{+}{5}\cdot \tfrac{1}{3}\cdot \underset{+}{e^{9-3\cdot\frac{1}{3}}}\,\left(\overset{2\;-\;1}{2 - \cancel{3}\cdot\tfrac{1}{\cancel{3}}}\right) > 0$$

$$g'(-1) = \underset{-}{5\cdot(-1)} \cdot \underset{+}{e^{9-3(-1)}} \underset{+}{(2 \cancel{-} \overset{+}{3}(-1))} < 0$$

Intervals : f is increasing in $(0, \frac{2}{3})$

f is decreasing in $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$

Local extrema : f has a local min at $x = 0$

$\xrightarrow{y\text{-coordinate}}$ $f(0) = 5(0)^2 (e^{9-0}) = 0$

$\Rightarrow$ $(0,0)$: local min point

f has a local max at $x = \frac{2}{3}$

$\xrightarrow{y\text{-coordinate}}$ $f(\frac{2}{3}) = 5(\frac{2}{3})^2 e^{9-3\cdot\frac{2}{3}} = \frac{20}{9} e^{9-2=7}$

$\Rightarrow$ $(\frac{2}{3}, \frac{20}{9}e^7)$: local max point

Practice Problems : For the following functions, use the 1st derivative test to determine if each critical point is a local min, max or neither.

1) $f(x) = \dfrac{x^3}{1-x^2}$

2) $f(x) = \dfrac{x}{\ln x}$

3) $h(x) = e^{x^2-4x}$

4) $g(x) = x^{2/3}(1-x^2)$

5) $g(x) = 2\sin^2 x - \sin x$ in $[0, 2\pi]$