Second Derivative. $f'' = (f')'$

Lecture 15. Feb 5.

In term 1, we defined the derivative to be the rate of change.

$f' \rightsquigarrow$ Rate of change in f.

$f'' = (f')' \rightsquigarrow$ Rate of change in f'

For example, if $x(t)$ is the position of a particle at time t, then

$x'(t) \rightsquigarrow$ Rate of change in position : $x'(t) = v(t) \rightsquigarrow$ velocity

$x''(t) \rightsquigarrow$ Rate of change in velocity : $x''(t) = v'(t) = a(t)$

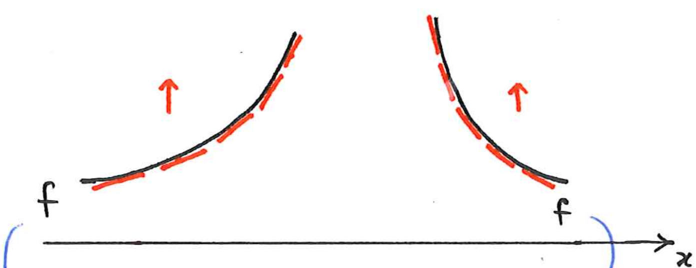
$\rightsquigarrow$ acceleration.

On the other we also know that at any point, the derivative of f is the slope of the tangent line to f at that point.

$f' \rightsquigarrow$ slope of the tangent line.

$f'' = (f')'$ $\rightsquigarrow$ Change in the slope of the tangent line.

f': slope; $(\,)'$: change

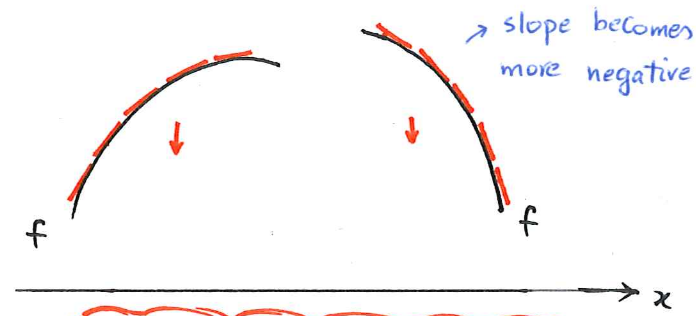


Slope of the tangent line keeps increasing

Slope still increases. It becomes less negative

$m_{tan} = f'$ is increasing

$\Longrightarrow (f')' = f''$ is positive



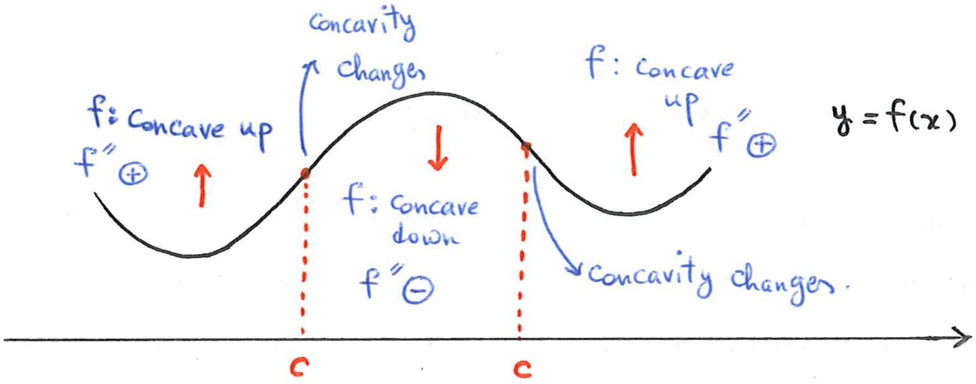
In both cases, slope of the tangent line keeps decreasing.

$m_{tan} = f'$ is decreasing

$\Downarrow$

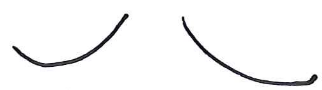
$(f')' = f''$ is negative.

Relationship between f'' and shape of f.



x=c is an inflection point, because the concavity of f changes at x=c.

Concavity of f :

Concavity determines how the curve of function is bent. If the curve opens upward , the graph is concave up.

If the curve opens downward , it is concave down.

Shape of f and sign of f''

- If $f(x)$ is concave up in an interval (a,b), then $f''(x) > 0$ for any x in (a,b).
- If $f(x)$ is concave down in (a,b), then $f''(x) < 0$ for all x in (a,b).

Vice versa :

- If $f''(x) > 0$ for any x in $(a,b) \Rightarrow f(x)$ is concave up in (a,b)
- If $f''(x) < 0$ for any x in $(a,b) \Rightarrow f(x)$ is concave down in (a,b).

Inflection point : $x = c$ is the inflection point of $f(x)$ if $f''(c) = 0$ and $f''(x)$ changes sign around c.

Now, in the examples, by determining the sign of $f''(x)$ we find the intervals where $f(x)$ is concave down / up.

<u>Example</u>. Find the intervals where f is concave down and concave up. Determine the inflection point if there's any.

$$f(x) = 2x^3 - 15x^2 + 24x - 7$$

$$f'(x) = 6x^2 - 30x + 24$$ ⟶ No need to find critical numbers, local max or min.

$$f''(x) = 12x - 30$$

$$f''(x) = 0 \Rightarrow 12x - 30 = 0 \Rightarrow x = \frac{30}{12} = \frac{5}{2}$$

possible inflection point. Need to check whether f'' changes sign around $5/2$.

Sign Chart for f'':

x	$-\infty$ test point $x=2$	$5/2$	test point $x=3$ $+\infty$
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$ ↓	inflection point	$\cup$ ↑

Test point for f'':

$x = 3 \Rightarrow f''(3) = 12 \times 3 - 30 = 6 > 0$

$x = 2 \Rightarrow f''(2) = 12 \times 2 - 30 = -6 < 0$

Summary
- In the interval $(-\infty, 5/2)$, f is concave down.
- In the interval $(5/2, \infty)$, f is concave up.
- At $x = 5/2$, f has an inflection point ⟶ $(5/2, -38/4)$