

MATH 110 Midterm 2, February 10th, 2016

Duration: 90 minutes

This test has 7 questions on 9 pages, for a total of 60 points.

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: SOLUTIONS

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	Total
Points:	9	10	10	10	12	5	4	60
Score:								

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - speaking or communicating with other examination candidates, unless otherwise authorized;
 - purposely exposing written papers to the view of other examination candidates or imaging devices;
 - purposely viewing the written papers of other examination candidates;
 - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Full-Solution Problems. In the following questions, justify your answers and show all your work. Unless otherwise indicated, simplification of answers are required.

9 marks

1. (a) Given the equation $2x^3 - 6xy^7 = -6$, find $\frac{dy}{dx}$.

$$2 \cdot 3x^2 - 6(1 \cdot y^7 + x \cdot 7y^6 y') = 0$$

$$6x^2 - 6y^7 - 42xy^6 y' = 0$$

$$y' = \frac{6x^2 - 6y^7}{42xy^6}$$

- (b) Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of $5 \text{ km}^2/\text{hr}$. How rapidly is the radius of the spill increasing when the area is 4 km^2 ?

$$A = \pi [r(t)]^2$$

$$\frac{dA}{dt} = 5 \text{ km}^2/\text{hr}$$

find $\frac{dr}{dt}$ when $A = 4 \text{ km}^2$?

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$\text{when } A = 4 = \pi r^2$$

$$r = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}}$$

$$\text{so } \frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} = \frac{5}{2\pi \frac{2}{\sqrt{\pi}}} = \boxed{\frac{5}{4\sqrt{\pi}} \text{ km/hr.}}$$

- (c) Find all critical numbers of $f(x) = x^{1/2} - x^{3/2}$.

$$f' = \frac{1}{2} x^{-1/2} - \frac{3}{2} x^{1/2} = \frac{1}{2} \left(\frac{1}{\sqrt{x}} - 3\sqrt{x} \right) = \frac{1}{2} \frac{1-3x}{\sqrt{x}}$$

$$f' = 0 \text{ when } 1-3x = 0$$

$$x = 1/3$$

f' is undefined at $x=0$

$$\text{so C. \#s are } \boxed{x=0, \frac{1}{3}}$$

0 marks

2. The curve given by the equation

$$(x^2 + y^2 - 1)^2 - 4(x^2 + y^2 - 2x + 1) = 0 \quad (1)$$

is known as a *cardioid*.(a) Find all the points $(1, y)$ that lie on the cardioid given above.

(b) One of the lines tangent to the cardioid given above at the points found in (a) has negative slope. Find the equation of this line.

$$(a) \quad (1, y) \quad (1 + y^2 - 1)^2 - 4(1 + y^2 - 2 + 1) = 0$$

$$y^4 - 4y^2 = 0$$

$$y = 0, \quad y^2 = 4$$

$$y = \pm 2$$

$$(1, 0), (1, 2), (1, -2)$$

$$(b) \quad 2(x^2 + y^2 - 1)(2x + 2yy') - 4(2x + 2yy' - 2) = 0$$

$$4x(x^2 + y^2 - 1) + 4y(x^2 + y^2 - 1)y' - 4(2x - 2) - 8yy' = 0$$

$$[4y(x^2 + y^2 - 1) - 8y]y' = 4(2x - 2) - 4x(x^2 + y^2 - 1)$$

$$y' = \frac{4(2x - 2) - 4x(x^2 + y^2 - 1)}{4y(x^2 + y^2 - 1) - 8y}$$

at $(1, 0)$ y' is undefined

$$\text{at } (1, 2) \quad y' = \frac{4 \cdot (0) - 4(1 + 4 - 1)}{8(1 + 4 - 1) - 16} = \frac{-16}{16} = -1 < 0$$

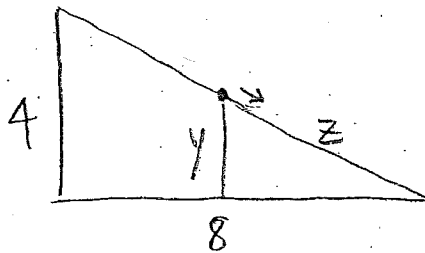
tangent line at $(1, 2)$ has equation $y - 2 = -(x - 1)$

$$\text{at } (1, -2) \quad y' = \frac{4 \cdot (0) - 4(1 + 4 - 1)}{-8(1 + 4 - 1) + 16} = \frac{-16}{-16} = 1 > 0$$

10 marks

3. Little Lizzie is sliding down the Great Green Run, her favourite straight slide at the local playground. The highest point of the slide is 4 metres above the ground and the slide is 8 metres long. When she is half way down the slide, she is losing elevation at 3 metres per second. How fast is she sliding down at that instant?

Make sure you organize your solution clearly, draw a sketch, label variables, etc.



Find z' when half way
given $y' = 3$ m/s

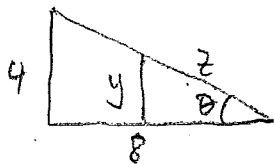
$$\frac{z}{y} = \frac{\sqrt{16+64}}{4}$$

$$z = \frac{\sqrt{80}}{4} y$$

$$z' = \frac{\sqrt{80}}{4} y'$$

$$\text{so } z' = \frac{\sqrt{80}}{4} \cdot 3 \text{ m/s}$$

Alternatively,



$$z = \frac{y}{\sin \theta}$$

$$z' = \frac{1}{\sin \theta} y'$$

$$\sin \theta = \frac{4}{\sqrt{80}}$$

$$\text{so } z' = \frac{\sqrt{80}}{4} \cdot 3 \text{ m/s}$$

0 marks

4. Two cylindrical swimming pools (with flat bottom) are being filled simultaneously with water, at exactly the same rate (measured in cubic metres per minute). The smaller pool has a radius of 5 metres and the height of the water in the smaller pool is increasing at a rate of 0.5 metres per minute. The larger pool has a radius of 8 metres. How fast is the height of the water increasing in the larger pool when the pool is a third full? You may use the fact that the volume V of a cylinder of base radius r and height h is $V = \pi r^2 h$.

Make sure you organize your solution clearly, draw a sketch, label variables, etc.



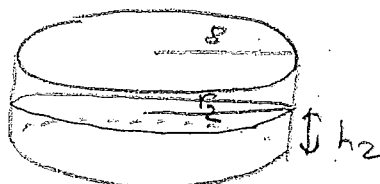
$$V_1 = \pi r_1^2 h_1$$

$$\frac{dV_1}{dt} = \pi \cdot 25 \frac{dh_1}{dt}$$

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

$$\text{so } \pi \cdot 25 \frac{dh_1}{dt} = \pi \cdot 64 \frac{dh_2}{dt}$$

$$\frac{dh_2}{dt} = \frac{25}{64} \frac{dh_1}{dt} = \boxed{\frac{25}{64} \cdot \frac{1}{2} \text{ m/min.}}$$



$$V_2 = \pi r_2^2 h_2$$

$$\frac{dV_2}{dt} = \pi \cdot 64 \cdot \frac{dh_2}{dt}$$

need to find $\frac{dh_2}{dt}$ given $\frac{dh_1}{dt} = \frac{1}{2}$

12 marks

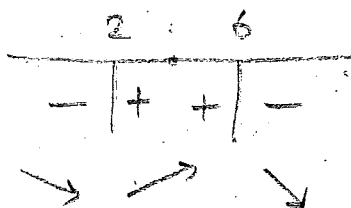
5. Consider a function f differentiable everywhere. Suppose the derivative f' satisfies all of the following conditions:

- $f'(x) = 0$ at $x = 2$ and $x = 6$ and $f'(x) \neq 0$ anywhere else,
- f' is increasing for $x < 5$ and decreasing for $x > 5$.

Based on the information above,

- determine all intervals where the function f is increasing and where it is decreasing.
- determine the x coordinates of all local maximum and minimum values of f (specify which values correspond to a maximum and which ones correspond to a minimum).

sign chart for f'



so f is increasing for $2 < x < 6$

decreasing for $x < 2$ and $x > 6$

f has a minimum at $x = 2$

maximum at $x = 6$

Let $g(x) = \frac{x^2 + 3}{x - 1}$.

- (a) Find intervals where the function is increasing and where it is decreasing.
 (b) Find all local maximum and minimum values of g , if they exist.

$$g' = \frac{2x(x-1) - (x^2+3) \cdot 1}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2}$$

$g' = 0$ when $x^2 - 2x - 3 = 0$

$$(x-3)(x+1) = 0$$

$$x = 3$$

$$x = -1$$

g' is undef. at $x = 1$

g also undef at $x = 1$.

sign chart for g'

-1	1	3	
+	-	-	+
→	↘	↘	→
max		min	

g is decreasing for $-1 < x < 3, x \neq 1$

g is increasing for $x < -1, x > 3$

max: $g(-1) = -2$

min: $g(3) = 6$

5 marks

6. (a) State the Mean Value theorem for a function f on an interval $[x_1, x_2]$. Make sure you state both the hypotheses and the conclusion of the theorem.

If f is continuous on $[x_1, x_2]$
differentiable on (x_1, x_2) then
there exists a number c in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

- (b) Let $f(x)$ be a function such that $f(1) = 2$ with the property that $f'(x) \geq 5$ on the interval $[1, 3]$. Use the Mean Value theorem to show that $f(3) \neq 8$.

$f(1) = 2$, $f'(x) \geq 5$ on $[1, 3]$, by MVT

$$\frac{f(3) - f(1)}{3 - 1} = f'(c) \geq 5$$

$$\frac{f(3) - 2}{2} \geq 5$$

$$f(3) \geq 10 + 2$$

$$\text{so } f(3) \neq 8.$$

4 marks

7. (a) If $p(x)$ is a polynomial of at least degree 2, show that between any two roots of p there is a root of p' .

Suppose $p(a)=0$ and $p(b)=0$ (so $x=a$ and $x=b$ are roots of $p(x)$), with $b>a$.

By MVT (which applies because $p(x)$ is differentiable everywhere) there must be a number c in (a,b) s.t.

$$\frac{p(b)-p(a)}{b-a} = p'(c)$$

Since $p(b)=0=p(a)$, it follows $p'(c)=0$

Thus $x=c$ is a root of p' .

- (b) Let $g(x)$ be a polynomial with at least three roots, show that g'' has at least one real root.

Suppose $g(x_1)=0$, $g(x_2)=0$, $g(x_3)=0$, with $x_1 < x_2 < x_3$

Then by MVT there must be a number c_1 in (x_1, x_2)

such that $g'(c_1)=0$ and a number c_2 in (x_2, x_3)

such that $g'(c_2)=0$.

Then by MVT applied to the function g' on the interval (c_1, c_2) , there must be a number

c_3 in (c_1, c_2) such that

$$0 = \frac{g'(c_2) - g'(c_1)}{c_2 - c_1} = g''(c_3)$$

So $x=c_3$ is a root of g'' .

