# MATH 110 Midterm 2, February 10th, 2016

**Duration: 90 minutes**

*This test has **7 questions** on **9 pages**, for a total of 60 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: __________ Last Name: SOLUTIONS

Student-No: __________ Section: __________

Signature: __________

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| Points: | 9 | 10 | 10 | 10 | 12 | 5 | 4 | 60 |
| Score: | | | | | | | | |

**Full-Solution Problems.** In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are required**.

9 marks

1. (a) Given the equation $2x^3 - 6xy^7 = -6$, find $\frac{dy}{dx}$.

$$2 \cdot 3x^2 - 6(1 \cdot y^7 + x \cdot 7y^6 y') = 0$$

$$6x^2 - 6y^7 - 42xy^6y' = 0$$

$$\boxed{y' = \frac{6x^2 - 6y^7}{42xy^6}}$$

(b) Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 5 km$^2$/hr. How rapidly is the radius of the spill increasing when the area is 4 km$^2$?

$$A = \pi [r(t)]^2$$

$$\frac{dA}{dt} = 5 \text{ km}^2/\text{hr}$$

find $\frac{dr}{dt}$ when $A = 4 \text{ km}^2$?

$$\frac{dA}{dt} = \pi \, 2r \frac{dr}{dt}$$

when $A = 4 = \pi r^2$

$$r = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}}$$

so $$\frac{dr}{dt} = \frac{\frac{dA}{dt}}{2\pi r} = \frac{5}{2\pi \frac{2}{\sqrt{\pi}}} = \boxed{\frac{5}{4\sqrt{\pi}} \text{ km/hr.}}$$

(c) Find all critical numbers of $f(x) = x^{1/2} - x^{3/2}$.

$$f' = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 3\sqrt{x}\right) = \frac{1}{2}\,\frac{1-3x}{\sqrt{x}}$$

$f' = 0$ when $1 - 3x = 0$, $x = 1/3$

$f'$ is undefined at $x = 0$

so C.#'s are $\boxed{x = 0, \frac{1}{3}}$

0 marks

2. The curve given by the equation

$$(x^2+y^2-1)^2-4(x^2+y^2-2x+1)=0 \tag{1}$$

is known as a *cardioid*.

(a) Find all the points $(1, y)$ that lie on the cardioid given above.

(b) One of the lines tangent to the cardioid given above at the points found in (a) has negative slope. Find the equation of this line.

(a) $(1,y)$ $(1+y^2-1)^2-4(1+y^2-2+1)=0$

$$y^4-4y^2=0$$

$$y=0, \quad y^2=4$$

$$y=\pm 2$$

$$\boxed{(1,0), \ (1,2), \ (1,-2)}$$

(b)

$$2(x^2+y^2-1)(2x+2yy')-4(2x+2yy'-2)=0$$

$$4x(x^2+y^2-1)+4y(x^2+y^2-1)y'-4(2x-2)-8yy'=0$$

$$[4y(x^2+y^2-1)-8y]y' = 4(2x-2)-4x(x^2+y^2-1)$$

$$y' = \frac{4(2x-2)-4x(x^2+y^2-1)}{4y(x^2+y^2-1)-8y}$$

at $(1,0)$ $y'$ is undefined

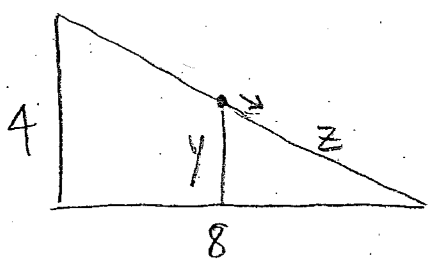
at $(1,2)$ $y' = \dfrac{4\cdot(0)-4(1+4-1)}{8(1+4-1)-16} = \dfrac{-16}{16} = -1 < 0$

tangent line at $(1,2)$ has equation $\boxed{y-2=-(x-1)}$

at $(1,-2)$ $y' = \dfrac{4\cdot(0)-4(1+4-1)}{-8(1+4-1)+16} = \dfrac{-16}{-16} = 1 > 0$

10 marks 3. Little Lizzie is sliding down the Great Green Run, her favourite straight slide at the local playground. The highest point of the slide is 4 metres above the ground and the slide is 8 metres long. When she is half way down the slide, she is losing elevation at 3 metres per second. How fast is she sliding down at that instant?

Make sure you organize your solution clearly, draw a sketch, label variables, etc.




Find $z'$ when half way given $y' = 3$ m/s

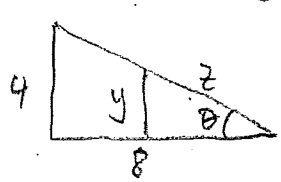
$$\frac{z}{y} = \frac{\sqrt{16+64}}{4}$$

$$z = \frac{\sqrt{80}}{4} y$$

$$z' = \sqrt{\frac{80}{4}}\, y'$$

so $\boxed{z' = \sqrt{\frac{80}{4}} \cdot 3 \quad m/s}$

Alternatively,




$$z = \frac{y}{\sin\theta}$$

$$z' = \frac{1}{\sin\theta} \cdot y'$$

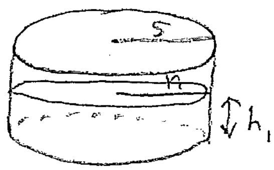
$\sin\theta = \frac{4}{\sqrt{80}}$ so $z' = \frac{\sqrt{80}}{4} \cdot 3$ m/s.

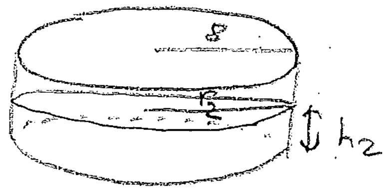
0 marks

4. Two cylindrical swimming pools (with flat bottom) are being filled simultaneously with water, at exactly the same rate (measured in cubic metres per minute). The smaller pool has a radius of 5 metres and the height of the water in the smaller pool is increasing at a rate of 0.5 metres per minute. The larger pool has a radius of 8 metres. How fast is the height of the water increasing in the larger pool when the pool is a third full? You may use the fact that the volume $V$ of a cylinder of base radius $r$ and height $h$ is $V = \pi r^2 h$.

   Make sure you organize your solution clearly, draw a sketch, label variables, etc.

$r_1 = 5$




$r_2 = 8$

need to find $\frac{dh_2}{dt}$ given $\frac{dh_1}{dt} = \frac{1}{2}$

$V_1 = \pi r_1^2 h_1$

$V_2 = \pi r_2^2 h_2$

$\frac{dV_1}{dt} = \pi \cdot 25 \frac{dh_1}{dt}$

$\frac{dV_2}{dt} = \pi \, 64 \cdot \frac{dh_2}{dt}$

$\frac{dV_1}{dt} = \frac{dV_2}{dt}$

so $\pi \cdot 25 \frac{dh_1}{dt} = \pi \, 64 \frac{dh_2}{dt}$

$\frac{dh_2}{dt} = \frac{25}{64} \frac{dh_1}{dt} = \boxed{\frac{25}{64} \cdot \frac{1}{2} \; m/min.}$

12 marks

5. Consider a function $f$ differentiable everywhere. Suppose the derivative $f'$ satisfies all of the following conditions:

   - $f'(x) = 0$ at $x = 2$ and $x = 6$ and $f'(x) \neq 0$ anywhere else,
   - $f'$ is increasing for $x < 5$ and decreasing for $x > 5$.

   Based on the information above,

   (a) determine all intervals where the function $f$ is increasing and where it is decreasing.

   (b) determine the $x$ coordinates of all local maximum and minimum values of $f$ (specify which values correspond to a maximum and which ones correspond to a minimum).

sign chart for $f'$

| | 2 | | 6 | |
|---|---|---|---|---|
| $-$ | | $+ \quad +$ | | $-$ |
| $\searrow$ | | $\nearrow$ | | $\searrow$ |

so $f$ is increasing for $2 < x < 6$
decreasing for $x < 2$ and $x > 6$

$f$ has a minimum at $x=2$
maximum at $x=6$

Let $g(x) = \dfrac{x^2+3}{x-1}$.

(a) Find intervals where the function is increasing and where it is decreasing.

(b) Find all local maximum and minimum values of $g$, if they exist.

$$g' = \frac{2x(x-1) - (x^2+3)\cdot 1}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2}$$

$g' = 0$ when $x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

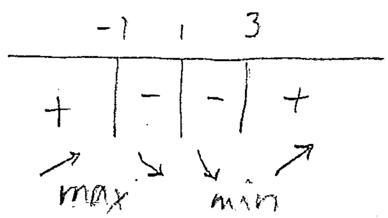
$x = 3$

$x = -1$

$g'$ is undef. at $x = 1$

$g$ also undef at $x = 1$.

sign chart for $g'$




$g$ is decreasing for $-1 < x < 3, x \neq 1$

$g$ is increasing for $x < -1, x > 3$

max: $g(-1) = -2$

min $g(3) = 6$

5 marks 6. (a) State the Mean Value theorem for a function $f$ on an interval $[x_1, x_2]$. Make sure you state both the hypotheses and the conclusion of the theorem.

If $f$ is continuous on $[x_1, x_2]$
differentiable on $(x_1, x_2)$ then
there exists a number $c$ in $(x_1, x_2)$ such that
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

(b) Let $f(x)$ be a function such that $f(1) = 2$ with the property that $f'(x) \geq 5$ on the interval $[1, 3]$. Use the Mean Value theorem to show that $f(3) \neq 8$.

$f(1) = 2$, $f'(x) \geq 5$ on $[1, 3]$, by MVT

$$\frac{f(3) - f(1)}{3-1} = f'(c) \geq 5$$

$$\frac{f(3) - 2}{2} \geq 5$$

$$f(3) \geq 10 + 2$$

so $f(3) \neq 8$.

4 marks

7. (a) If $p(x)$ is a polynomial of at least degree 2, show that between any two roots of $p$ there is a root of $p'$.

Suppose $p(a)=0$ and $p(b)=0$ (so $x=a$ and $x=b$ are roots of $p(x)$), with $b>a$.
By MVT (which applies because $p(x)$ is differentiable everywhere) there must be a number $c$ in $(a,b)$ s.t.

$$\frac{p(b)-p(a)}{b-a}=p'(c)$$

Since $p(b)=0=p(a)$, it follows $p'(c)=0$
Thus $x=c$ is a root of $p'$.

(b) Let $g(x)$ be a polynomial with at least three roots, show that $g''$ has at least one real root.

Suppose $g(x_1)=0$, $g(x_2)=0$, $g(x_3)=0$, with $x_1<x_2<x_3$
Then by MVT there must be a number $c_1$ in $(x_1,x_2)$ such that $g'(c_1)=0$ and a number $c_2$ in $(x_2,x_3)$ such that $g'(c_2)=0$.
Then by MVT applied to the function $g'$ on the interval $(c_1,c_2)$, there must be a number $c_3$ in $(c_1,c_2)$ such that

$$0=\frac{g'(c_2)-g'(c_1)}{c_2-c_1}=g''(c_2)$$

So $x=c_2$ is a root of $g''$.