HW 3, Due on Friday.

Lecture 16, Feb 7.

Quick Summary :

Goal: Get info about the shape of a function $f(x)$

From $f(x)$ $\rightarrow$

- Domain
- intercepts
- Asymptotes (Next week)

From $f'(x)$ $\rightarrow$

sign of f' is related to increase or decrease in f.

- f' ⊕ $\Downarrow$ f increasing
- f' ⊖ $\Downarrow$ f decreasing

local extrema

1st derivative test:

- f goes from $\searrow$ to $\nearrow$ $\Rightarrow$ local min
  or f' " " ⊖ to ⊕
- f goes from $\nearrow$ to $\searrow$ $\Rightarrow$ local max
  or f' " " ⊕ to ⊖

From $f''(x)$ $\rightarrow$

sign of f'' is related to Concavity of f.

$\downarrow$

bending of the curve

- f'' ⊕ $\Downarrow$ Concave up
- f'' ⊖ Concave down

inflection point is where f'' change sign.

Examples • Determine the intervals in which f decreases/increases. Find local extrema. Determine concavity and inflection point.

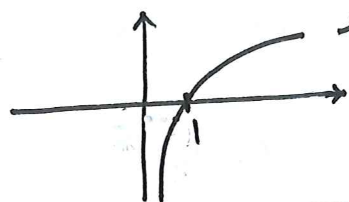
(1)

WW 2.6. $f(x) = x \ln x$

Domain.

Recall : $\ln$ ▨ is defined when ▨ is positive

$(\ln x)' = \frac{1}{x}$



$\ln 1 = 0$

$\ln e = 1$

Domain:

$\boxed{x > 0}$

Critical numbers:

$f'(x) = 1 \cdot \ln x + \cancel{x}' \cdot \frac{1}{\cancel{x}} = \ln x + 1$

For concavity $\leftarrow f''(x) = \frac{1}{x}$

$f'(x) = 0$ or $f'(x)$ NOT defined.

when x is 0 or negative

$\ln x + 1 = 0$

$\ln x = -1$

$\cancel{e}^{\cancel{\ln} x} = e^{-1}$

$\Rightarrow \boxed{x = e^{-1} = \frac{1}{e}}$

Sign chart for f':

test for $\frac{1}{e^2} < \frac{1}{e}$. test for $e > \frac{1}{e}$

x	0		$\frac{1}{e} = e^{-1}$		$+\infty$
$f'(x)$	∞	$-$	0	$+$	
$f(x)$		$\searrow$	local min at $x = \frac{1}{e}$	$\nearrow$	

f and f' are NOT defined for $x = 0$ and negative x values.

⊛ Note that $\frac{1}{e} \simeq \frac{1}{2.71\ldots} \simeq \frac{1}{3}$

so $e > \frac{1}{e}$ $\xrightarrow{\text{test } f'}$

and $\frac{1}{e^2} \simeq \frac{1}{9} < \frac{1}{e}$

$$\begin{cases} f'(x) = \ln x + 1 \\ f'(e) = \ln e + 1 = 1 + 1 = 2 > 0 \\ f'(e^{-2}) = \ln e^{-2} + 1 = -2 + 1 = -1 < 0 \end{cases}$$

Summary from f':
- f is decreasing in $(0, \frac{1}{e})$
- f is increasing in $(\frac{1}{e}, \infty)$
- f has a local min at $x = \frac{1}{e} = e^{-1}$

and the minimum value is $f(\frac{1}{e}) = \frac{1}{e} \ln \frac{1}{e} = \frac{1}{e} \ln e^{-1} = -\frac{1}{e}$ $\Rightarrow (\frac{1}{e}, -\frac{1}{e})$ local min point

Sign chart for f'':

$f''(x) = \frac{1}{x} = 0$ NEVER b/c $1 \neq 0$

$\frac{1}{x} = \frac{+}{+} > 0$

x	0	all positive x	
$f''(x)$	∞	$+$	$+$
$f(x)$		$\cup$	$\cup$

Summary from f'': f is always concave up and it has no inflection point b/c f'' does NOT change sign.

(2) $f(x) = \dfrac{x}{x^2-1}$

$x^2 - 1 = 0$

$x^2 = 1$

$\Downarrow$

$x = 1 \quad , x = -1$

Domain: where $x^2 - 1 \neq 0$

$(x-1)(x+1) = 0 \Rightarrow x = 1 \quad , \quad x = -1$

$\boxed{\text{Domain: } \quad x \neq 1 , x \neq -1}$

f':

$$f'(x) = \frac{1 \cdot (x^2-1) \quad - x \quad \cdot \quad 2x}{(x^2-1)^2}$$

$$= \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$$

$\downarrow$

$f'(x)$ is 0 when

$-x^2 - 1 = 0$

$-x^2 = 1$

$x^2 = -1$

NEVER

$\downarrow$

$f'(x)$ is NOT defined

$(x^2-1)^2 = 0$

$x^2 - 1 = 0$

$\Downarrow$

$\underbrace{x = 1 , x = -1}_{}$

NOT in the domain

$\Rightarrow$ NOT critical numbers

But they come in sign chart.

$\boxed{\text{Sign chart for } f'}$:

		test $x=-2$		test $x=0$		test $x=2$	
x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$-$	∞	$-$	∞	$-$	
$f(x)$		↘		↘		↘	

everywhere decreasing

⇓

NO local min/max

$$f'(x) = \frac{-x^2-1}{(x^2-1)^2}$$

$$f' = \frac{-}{+} < 0$$

Intervals: $\underbrace{(-\infty,-1)\ ,\ (-1,1)\ ,\ (1,\infty)}_{\text{decreasing}}$

$$f'(x) = \frac{-x^2-1}{(x^2-1)^2}$$

$$f''(x) = \frac{-2x(x^2-1)^2 - (-x^2-1)\cdot \overbrace{2\cdot 2x(x^2-1)}^{\text{chain rule}}}{(x^2-1)^4}$$

Do NOT expand

Do NOT distribute

LOOK For Factoring

$$f''(x) = \frac{2x\cancel{(x^2-1)}\left(\overbrace{-(x^2-1)+2(x^2+1)}^{-x^2+1+2x^2+2}\right)}{(x^2-1)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}$$

Exercise: Sign chart for f''.

Sign chart for f'':

$$2x(x^2+3)=0$$

$f''(x)=0$ $\qquad \boxed{x=0} \qquad x^2+3=0 \Downarrow x^2=-3$ NEVER

f'' is Not defined at $x=1$ & $x=-1$

		$x=-2$		$x=-\frac{1}{2}$		$x=\frac{1}{2}$		$x=2$	
x	$-\infty$		-1		0		1		$+\infty$
$f''(x)$		$-$	∞	$+$	0	$-$	∞	$+$	
$f(x)$		$\cap$		$\cup$		$\cap$		$\cup$	

$$f''(x)=\frac{2x(x^2+3)}{(x^2-1)^3}$$

$x^2+3 \rightarrow +$

$$f''(2)=\frac{2\cdot 2\cdot +}{(4-1)^3}=\frac{+}{+}$$

$$f''\left(\tfrac{1}{2}\right)=\frac{2\cdot\frac{1}{2}\cdot +}{\left(\frac{1}{4}-1\right)^3}=\frac{+}{-}$$

$$f''\left(-\tfrac{1}{2}\right)=\frac{2\cdot -\frac{1}{2}\cdot +}{\left(\frac{1}{4}-1\right)^3}=\frac{-}{-}$$

$$f''(-2)=\frac{2\cdot(-2)\cdot +}{(4-1)^3}=\frac{-}{+}$$

Summary from f'' :

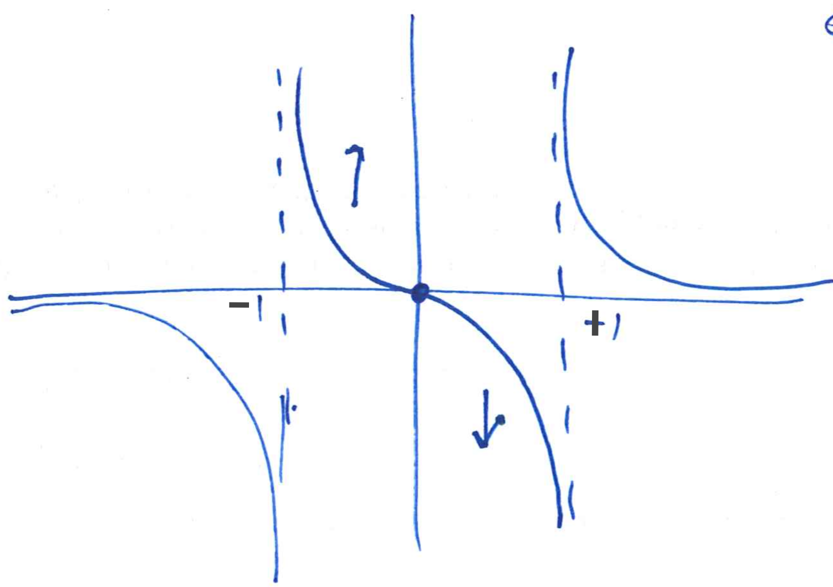
- f is concave up in $(-1,0)$ and $(1,\infty)$
- " " " down in $(-\infty,-1)$ and $(0,1)$
- inflection point: f'' changes sign

arround $\cancel{x=-1}$, $\boxed{x=0}$, $\cancel{x=1}$

f is NOT defined at $x=1$, $x=-1$

f has an inflection point only at $x=0$

y-coordinate : $f(0)=0 \Rightarrow (0,0)$



everywhere decreasing