Implicit Diff

(1) (f) $e^{\sin x} + e^{\cos y} = x$

$$e^{\sin x} \cdot \cos x + e^{\cos y} \cdot (-\sin y) \cdot y' = 1$$

$$\Rightarrow e^{\sin x} \cdot \cos x - 1 = \left(\sin y \, e^{\cos y}\right) y'$$

$$\Rightarrow y' = \frac{\cos x \, e^{\sin x} - 1}{\sin y \; e^{\cos y}}$$

(b) $\sin\left(\frac{x}{y}\right) = \frac{1}{2}$

$$\cos\left(\frac{x}{y}\right)\left(\frac{1 \cdot y - x \cdot y'}{y^2}\right) = 0$$

Solve for $y'$

---

2 (d) $x^2 y + y^4 = 4 + 2x$ $\qquad$ $x$ - intercept $\rightsquigarrow$ $\underline{y = 0}$

$$2xy + x^2 y' + 4y^3 y' = 2$$

$$x^2 \cdot 0 + 0 = 4 + 2x$$

$$0 = 4 + 2x$$

$$-4 = 2x$$

$$\boxed{-2 = x}$$

$$\Rightarrow y' = \frac{2 - 2xy}{x^2 + 4y^3}$$

$$= \frac{2 - \cancel{2 \times (-2) \times 0}^{\,0}}{(-2)^2 + \cancel{4(0)^3}} = \frac{2}{4} = \frac{1}{2}$$

$(-2, 0)$

$(0, 0)$

tangent line at $(a, f(a))$ is $\boxed{y - f(a) = f'(a)(x-a)}$

$$y - 0 = \frac{1}{2}(x-(-2))$$

$$\boxed{y = \frac{1}{2}(x+2)}$$

---

5 $y'$ at $(0,1) = -1$ , also $(0,1)$ is on the curve

$x^3 - 6xy - ky^3 = a$ $\qquad k, a$ ?

$$3x^2 - 6(y + xy') - k\,3y^2 y' = 0$$

$$3x^2 - 6y - 6xy' - 3ky^2y' = 0$$

$$y' = \frac{3x^2 - 6y}{6x + 3ky^2}$$

$$-1 = \frac{3(0)^2 - 6(1)}{6(0) + 3k(1)^2} = \frac{-6}{3k}$$

$$\Rightarrow \quad -1 = \frac{-6}{3k} \quad \Rightarrow \quad -3k = -6 \quad \Rightarrow \boxed{k = \frac{-6}{-3} = 2}$$

plug in (0,1) into the original curve

$$\cancel{(0)^3} - 6\cdot\cancel{0}\cdot 1 - 2(1)^3 = a$$

$$\boxed{-2 = a}$$

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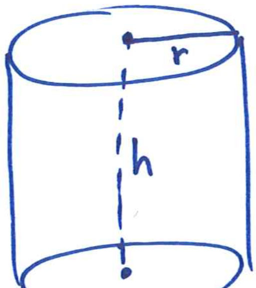
## Related Rates

$$\underset{\text{constant} \,=\, 600\pi \text{ cm}^2}{A} = 2\pi r\cdot h$$

(2).




$r'(t) = 4 \text{ cm/sec}$

$h'(t) = ?$ when $\boxed{r = 10 \text{ cm}}$

Do NOT substitute

$$\underbrace{A'}_{0} = 2\pi\Big(\underbrace{r'}_{4}\, h + \underbrace{r}_{10}\, \underbrace{h'}_{?}\Big)$$

I should find h

$$\boxed{\begin{array}{l} A = 2\pi r h \\ 600\pi = 2\pi \times 10 \times h \Rightarrow h = \dfrac{60\cancel{0}\cancel{\pi}}{2\cancel{\pi}\times 1\cancel{0}} = 30 \text{ cm} \end{array}}$$

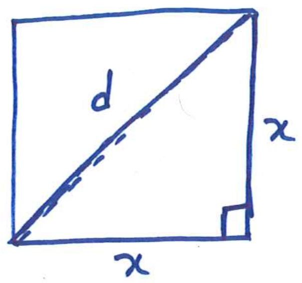
$$0 = 2\pi(4\times 30 + 10\times h')$$

$$0 = 2\pi(120 + 10h') \Rightarrow 120 + 10h' = 0$$

$$\Rightarrow 120 = -10h'$$

$$\Rightarrow h' = -12 \text{ cm/sec}$$

(8)




$d'(t) = 4 \ m/min$

$A'(t) = ?$

$d = 14 \ m$

$A = x^2$ → Use Pythagorean to relate $x$ and $d$.

$d^2 = x^2 + x^2$

$d^2 = 2x^2$

$$\boxed{d = \sqrt{2}\, x}$$

$x = \dfrac{d}{\sqrt{2}}$

$A = \left(\dfrac{d}{\sqrt{2}}\right)^2$

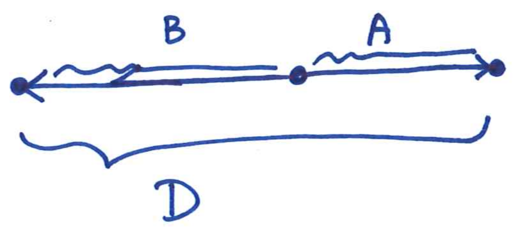
$A(t) = \dfrac{d(t)^2}{2}$

$A' = \frac{1}{2}(2dd') = dd'$

$A' = 14 \times 4 = 56 \ m^2/min$

---

(9)




$A' = 2 \ mm/sec$

$B' = 3 \ mm/sec$

$D' = ?$

$D = A + B$

$D' = A' + B' = 2 + 3 = 5 \ mm/sec$

# MVT and Rolle

(1) (d) $f(x) = \frac{1}{x^2}$ $[-2,2]$

must be

cont is in $[-2, 2]$ ?

and diff'able in $(-2, 2)$ ?

At $x = 0$ $f(x)$ is NOT defined

So $f$ is not cont's on the whole $[-2,2]$

(2) (d) $f(x) = 8 + e^{-3x}$ on $[-2,3]$

$f$ is cont's and diff'able everywhere including $[-2,$

By MVT there is "c" such that

$$f'(c) = \frac{f(3) - f(-2)}{3 - (-2)}$$

$f'(x) = -3e^{-3x}$

$$-3e^{-3c} = \frac{8 + e^{-9} - (8 + e^{6})}{5}$$

$$-3e^{-3c} = \frac{e^{-9} - e^{6}}{5}$$

$$\ln \downarrow \quad e^{-3c} = \frac{e^{-9} - e^{6}}{-15} \quad \downarrow$$

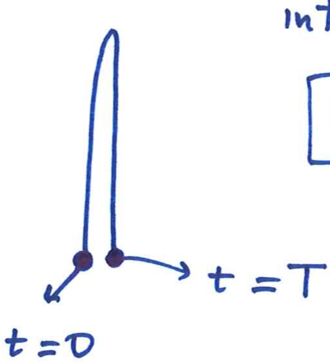
$$\ln e^{-3c} = \ln\left(\bigcirc\right)$$

$$-3c = \ln\left(\bigcirc\right)$$

$$c = \frac{\ln\left(\bigcirc\right)}{-3}$$

---

(10)




interval in time $[0, T]$

$\boxed{v(t) = 0}$ ?

$h(t)$ = height of ball is a cont.s function in $[0,T]$ and it.s diff. able in $(0,T)$.

$v(t) = h'(t) = ?$ $\qquad h(0) = h(T)$

By Rolle's Theorem there is a c in $[0,T]$ such that $\underline{h'(c) = 0}$

$v(c) = 0$

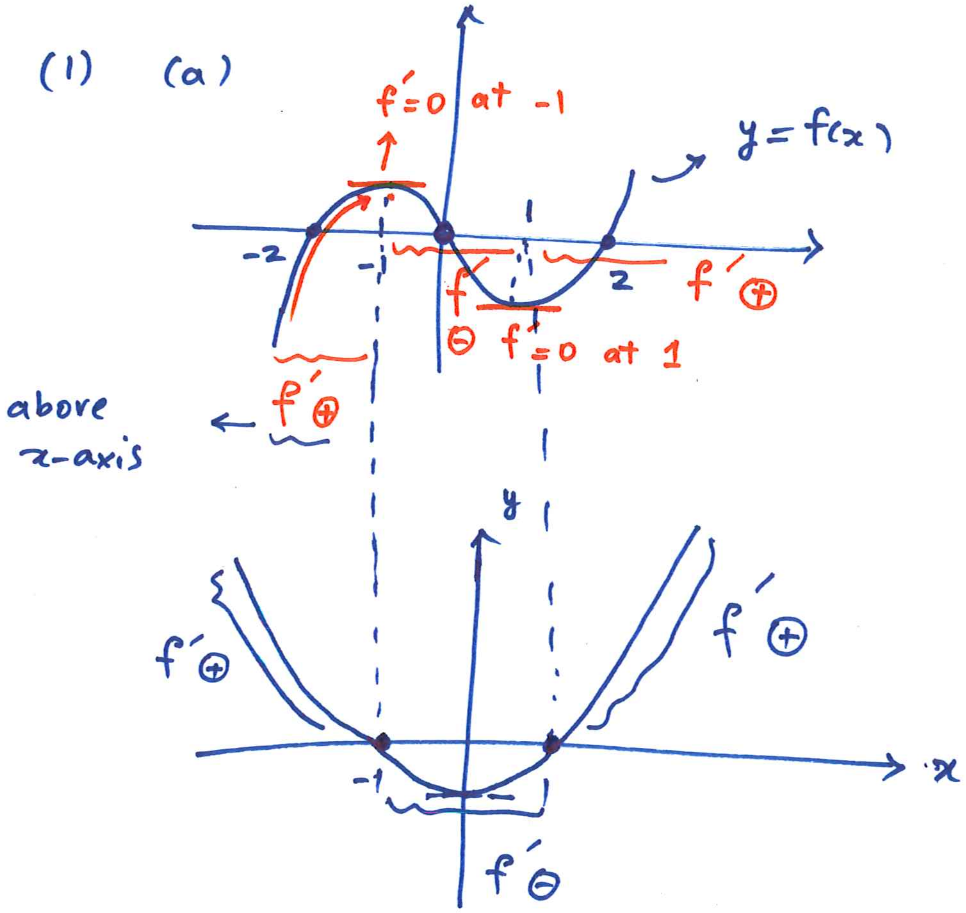
local extrema . . .

$x = 0$ INF point

$f'(0)$ has a horizontal li

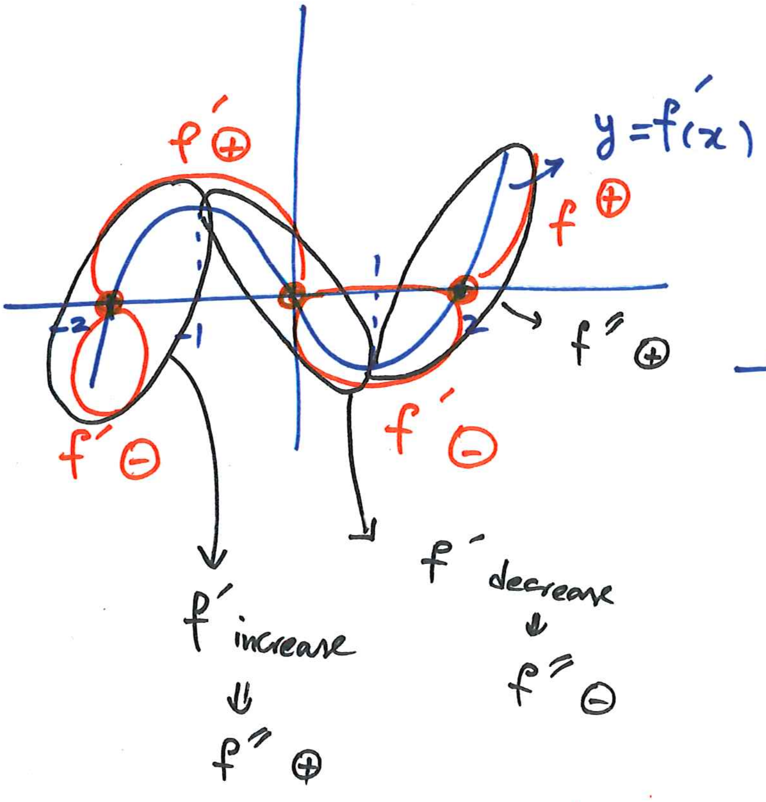
$f''(0) = 0$

(1) (a)




$y = f'(x)$ .

Change the graph in part (a) to be the graph of $f'(x)$.




→ Critical numbers of f

→ intervals of decrease and increase in f

→ intervals concave up/down in f

→ inflection points of f

Critical numbers $f'(x) = 0 \longrightarrow f'$ crosses the $x$-axis

$x = -2 \ , \ 0 \ , \ 2$

above $x$-axis

Increase / decrease : $f'$ ⊕ $\rightsquigarrow$ $f$ increases

$f'$ ⊖ $\rightsquigarrow$ $f$ decreases.

below $x$-axis

---

$f$ increases in $(-2, 0)$ and $(2, \infty)$

$f$ decreases in $(-\infty, 2)$ and $(0, 2)$

→ local min/max by a sign chart

---

Concavity : $f''$ ⊕ when $f'$ increases

$f''$ ⊖ when $f'$ decreases

$f$ is concave up in $(-\infty, -1)$ and $(1, \infty)$

$f$ " " down in $(-1, 1)$

| | | -1 | | 1 | |
|---|---|---|---|---|---|
| $f''$ | + | | — | | + |

INF points

(4) Domain : $\mathbb{R}$

(a) $f'(x) = 4x^{-\frac{1}{3}} - 4$

$$= \frac{4}{x^{1/3}} - 4 = \frac{4 - 4x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$f'(x) = 0$

$\Downarrow$

$$4 - 4x^{\frac{1}{3}} = 0$$

$$4 = 4x^{\frac{1}{3}}$$

$$1 = x^{\frac{1}{3}}$$

$$1 = \sqrt[3]{x}$$

$$\boxed{1 = x}$$

$f'(x)$ ND .

$\Downarrow$

$$x^{\frac{1}{3}} = 0$$

$\Downarrow$

$$\boxed{x = 0}$$ ✓

in the domain

test $-1$ | 0.5 | test 2

| $x$ | $-\infty$ | $\uparrow$ | 0 | $\nearrow$ | 1 | $\nearrow$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $f'$ | | $-$ | $\infty$ | $+$ | 0 | $-$ | |
| $f$ | | $\searrow$ | local min | $\nearrow$ | local max | $\searrow$ | |

$$f'(2) = \frac{4 - 4\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{-}{+}$$

$$f'\left(\tfrac{1}{2}\right) = \frac{4 - 4\sqrt[3]{\frac{1}{2}}}{\sqrt[3]{\frac{1}{2}}} = \frac{+}{+}$$

$$f'(-1) = \frac{4 - 4\sqrt[3]{-1}}{\sqrt[3]{-1}} = \frac{+}{-}$$

(6) (a) $f(x) = 4x^4 - x^3 + 2$

$$f'(x) = 16x^3 - 3x^2$$

$$f''(x) = 48x^2 - 6x = 0$$

$$= 6x(8x - 1) = 0$$

$\longrightarrow 6x = 0 \Rightarrow x = 0$

$\longrightarrow 8x - 1 = 0 \Rightarrow x = \frac{1}{8}$

test $(-1)$    test $\frac{1}{10}$    test $1$

| $x$ | $\uparrow$ | $0$ | $\uparrow$ | $\frac{1}{8}$ | $\nearrow$ |
|---|---|---|---|---|---|
| $f''$ | $+$ | $0$ | $-$ | $0$ | $+$ |
| $f$ | $\cup$ | INF | $\cap$ | INF | $\cup$ |

$$f''(1) = 6(8-1) = \oplus$$

$$f''\left(\tfrac{1}{10}\right) = \frac{6}{10}\left(\frac{8}{10} - 1\right) = \ominus$$

$$-\frac{2}{10}$$

$$f''(-1) = 6(-1)(8(-1)-1)$$

$$= -6(-9) = \oplus$$

---

(7) (a) $f(x) = x^4 - 2ax^2 + b$

info: $f$ has a critical point at $(2,5)$.

$\boxed{f' \text{ at } x=2 \text{ is } 0}$

$$f'(x) = 4x^3 - 2a(2x)$$

$$f'(2) = 0 \Rightarrow 4(2)^3 - 2a(2\times 2) = 0$$

$$32 - 8a = 0 \Rightarrow \boxed{a = \frac{32}{8} = 4}$$

also

$$f(2) = 5 \underset{a=4}{\Rightarrow} f(2) = 2^4 - 8\times(2)^2 + b = 5$$

$$16 - 32 + b = 5$$

$$-16 + b = 5$$

$$\boxed{b = 5 + 16 = 21}$$

---

(1) Asymptotes :

(c) $y = \dfrac{x+3}{x^2-9} = \dfrac{x+3}{(x-3)(x+3)}$ candidates for VA = 3, -3

VA :

$$\lim_{x \to 3} \frac{x+3}{x^2-9} = \lim_{x \to 3} \frac{1}{x-3} = \frac{1}{0}$$

$\to \lim_{x \to 3^+} \frac{1}{x-3} = \frac{1}{0^+} = \infty$

$\to \lim_{x \to 3^-} \cdots = -\infty$

$$\lim_{x \to -3} \frac{x+3}{x^2-9} = \lim_{x \to -3} \frac{1}{x-3} = \frac{1}{-6}$$

$\Rightarrow$ $\boxed{x = 3 \text{ is the only VA}}$

HA :

$$\lim_{x \to \infty} \frac{x+3}{x^2-9} = \lim_{x \to \infty} \frac{\cancel{x}(1 + \overset{0}{\cancel{3/x}})}{x^{\cancel{2}}(1 - \underset{0}{\cancel{9/x^2}})} = \lim_{x \to +\infty} \frac{1}{x} = 0$$

Similar for $x \to -\infty$ $\Rightarrow$ $\boxed{y = 0 \text{ is HA}}$

(f) $y = \frac{1}{e^x}$

VA is when $e^x = 0$, it NEVER Happens $\Rightarrow$ NO VA

HA : $\lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{e^\infty} = \frac{1}{\text{very large}} = 0 \Rightarrow y = 0$ is the HA.

$\Downarrow$

still ~~an~~ a HA but one-sided

However:

$$\lim_{x\to-\infty} \frac{1}{e^x} = \frac{1}{e^{-\infty}} = e^\infty = \infty$$

---

(2) $\lim_{x\to\infty} 2x^2+1 = \lim_{x\to\infty} 2x^2\left(1+\cancel{\frac{1}{2x^2}}\right) = \lim_{x\to\infty} 2x^2 = \infty$

$0$

also $\lim_{x\to-\infty} 2x^2+1 = \infty$.

## True / False

(1) If c is a critical number, then (c, f(c)) is either a local max or min. F

(2) If c is NOT a critical number, then (c, f(c)) can NOT be a local min/max. T

(3) Function $f(x) = |x|$ has no critical points. F

(4) If $f''(a) = 0$, then f has an inflection point at "a". F

(5) If f(x) is cont's on [a,b] and f(a) = f(b), then there's a point c in (a,b) at which f has a horizontal tangent line. F

(6) If $f''(x) > 0$ for all x in (a,b), then f is increasing in (a,b). F

(7) If f is concave up in (a,b), then its derivative $f'$ is increasing. T

(8) The second derivative of every linear function is zero. T

(9) For the function $f(x) = \frac{1}{x^2}$, MVT can be applied on (0,2). F

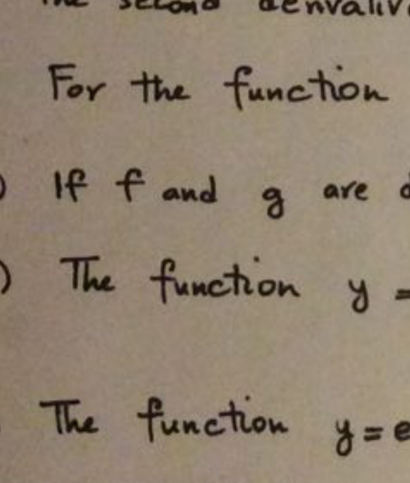
(10) If f and g are differentiable and increasing then f+g is increasing. T

(11) The function $y = \frac{x^2+4x+4}{x^2-x-6}$ has a VA at $x = 3$ and HA at $y = 1$. T

(12) The function $y = e^x$ has NO HA because $\lim_{x\to\infty} e^x = \infty$ F

## Multiple Choice

(1) Given the following graph




a) $f'(A) > 0$, $f'(B) > 0$, $f'(C) > 0$

b) $f'(A) > 0$, $f''(A) < 0$, $f''(C) > 0$

c) $f'(A) > 0$, $f''(A) < 0$, $f'(C) < 0$

d) $f''(A) > 0$, $f'(B) = 0$, $f''(B) < 0$

(2) MVT guarantees the existence of a number "c" on the graph of $y = \sqrt{x}$ between (0,0) and (4,2). What are the coordinates of this point?

(a) (2,1) (b) (1,1) (c) $(2, \sqrt{2})$ (d) $(\frac{1}{2}, \frac{1}{\sqrt{2}})$

(3) $f(x) = e^{-2x} - 2x$ is

(a) always increasing

(b) always decreasing

(c) has a local min at $x = 0$

(d) always concave down.