total = 33 + 2 bonus

Homework 4, MATH 110-001

Due date: Friday, March 16, 2018 (in class)

Hand in full solutions to the questions below. Make sure you justify all your work and include complete arguments and explanations. Your answers must be clear and neatly written, as well as legible (no tiny drawings or micro-handwriting please!). Your answers must be stapled, with your name and student number at the top of each page.

1. Hand-sketch the following functions using the steps covered in lectures (Be sure to highlight all important points/trends in your sketch).

   (a) $f(x) = \dfrac{x^2}{x^2+3}$

   (b) $f(x) = x^2 e^x$

2. The family of Normal (or Gaussian) curves play a very important roll in science, forestry, medicine, and business. You may have heard of the so-called Bell curve. These curves are defined by two parameters, the mean μ and the standard deviation $\sigma > 0$, and the curve satisfies the function

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

   (Hint: The only variable for this function is x. You should treat μ and σ as constants.)

   (a) What is the domain of $y = N(x)$? What is the range of $y = N(x)$?

   (b) Find and classify the following on the entire domain of $y = N(x)$

   - local minimum and local maximum
   - global minimum and global maximum

   (c) Now assume $\mu = 0$ and $\sigma = 1$, find the intervals for concavity of $N(x)$ and its inflection points.

   (d) Sketch the Bell curve for $\mu = 0$ and $\sigma = 1$.

3. In part (a) and (b) below, a set of conditions on a function f is given. For each part, sketch the graph of a function f that satisfies all the given conditions. (Making a combined summary chart is a helpful strategy.)

   (a) Function f satisfies

   - f is continuous and differentiable everywhere and $\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = -\infty$,
   - $f'(0) = f'(2) = f'(4) = 0$,
   - $f(0) = 0, f(2) = -2, f(4) = 4$,
   - $f'(x) > 0$ for $x < 0$ or $2 < x < 4$,
   - $f'(x) < 0$ for $0 < x < 2$ or $x > 4$,
   - $f''(x) > 0$ for $1 < x < 3$,
   - $f''(x) < 0$ for $x < 1$ or $x > 3$.

(b) Function f satisfies

- Domain of f is all non-negative numbers except $x = 3$, i.e. Domain $= \{x \geq 0;\ \text{such that } x \neq 3\}$,
- f has a root at $x = 0$ and $\lim_{x \to \infty} f(x) = -1$, $\lim_{x \to 3^-} f(x) = \infty$, $\lim_{x \to 3^+} f(x) = -\infty$,
- $f(x)$ is negative on the interval $(0, 2) \cup (3, \infty)$ and positive on the interval $(2, 3)$,
- $f(x)$ is differentiable on the entire domain with critical points at $x = 1, 5$,
- $f'(x) > 0$ on the interval $(1, 3) \cup (3, 5)$ and $f'(x) < 0$ on $(0, 1) \cup (5, \infty)$,
- The second derivative of $f(x)$ exists on the entire domain,
- $f''(x) > 0$ on the interval $(0, 3) \cup (8, \infty)$ and $f''(x) < 0$ on the interval $(3, 8)$.

Bonus: Extend your graph in part (b) above to all of real numbers $\mathbb{R}$ such that

(i) The graph becomes symmetric with respect to the y-axis. (This is called an *even* function.)

(ii) The graph becomes symmetric with respect to the origin. (This is called an *odd* function.)

(1) Each one : 6 pts

(a) $f(x) = \dfrac{x^2}{x^2+3}$

Domain : $x^2+3=0 \Rightarrow x^2=-3$ NEVER $\Rightarrow$ Domain $=\mathbb{R}$

intercepts : $\begin{cases} x\text{-int} : \; y=0 = \dfrac{x^2}{x^2+3} \Rightarrow x^2=0 \Rightarrow x=0 \Rightarrow (0,0) \\ y\text{-int} : \; x=0 \Rightarrow \dfrac{0}{0+3} = 0 \end{cases}$

VA : NO VA

HA : $\lim\limits_{x\to\infty} \dfrac{x^2}{x^2+3} = \dfrac{\infty}{\infty} \rightsquigarrow$ l'Hopital $\lim\limits_{x\to\infty} \dfrac{\cancel{2x}}{\cancel{2x}} = 1 \Rightarrow y=1$

Similarly : $\lim\limits_{x\to-\infty} \dfrac{x^2}{x^2+3} = 1$

$f'(x) = \dfrac{2x(x^2+3)-2x(x^2)}{(x^2+3)^2} = \dfrac{6x}{\underbrace{(x^2+3)^2}_{\text{always } \oplus}}$ → defined everywhere

→ $f'(x)=0 \Rightarrow 6x=0 \Rightarrow x=0$

x	test -1	0	test 1
f'	$-$	0	$+$
f	↘	local min $(0,0)$	↗

$f''(x) = \dfrac{6(x^2+3)^2 - 6x\left(2(x^2+3).2x\right)}{(x^2+3)^4} = \dfrac{6(x^2+3)\left[x^2+3-4x^2\right]}{(x^2+3)^4} = \dfrac{6(3-3x^2)}{\underbrace{(x^2+3)^3}_{\text{always } \oplus}}$

$f''(x)=0 \Rightarrow 3-3x^2=0 \Rightarrow x^2=1 \Rightarrow x=1 \;, x=-1$

x	-2	-1	0	1	2
f''	$-$	0	$+$	0	$-$
f	$\frown$	INF	$\smile$	INF	$\frown$

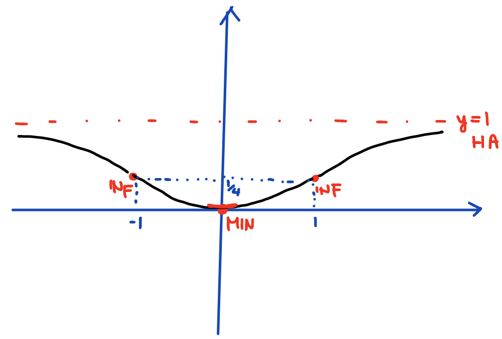
$f''(2) = \dfrac{6(3-12)}{+} < 0$

$f''(0) = \dfrac{6.(3-0)}{+} > 0$

$f''(-2) < 0$

$f(-1) = \dfrac{(-1)^2}{(-1)^2+3} = \dfrac{1}{4} = f(1)$

x		-1		0		1	
f'	$-$ ↘		$-$ ↘	0	$+$ ↗		$+$ ↗
f''	$-$ $\cap$	0	$+$ $\cup$		$+$ $\cup$	0	$-$ $\cap$
f		INF		MIN		INF	



(b) $f(x) = x^2 e^x$

Domain $= \mathbb{R}$, y-int $\xrightarrow{x=0} y = 0e^0 = 0 \times 1 = 0$ $\longrightarrow (0,0)$

x-int $\longrightarrow x^2 e^x = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

NO VA.

HA: $\lim\limits_{x\to\infty} x^2 e^x = \infty \cdot e^\infty = \infty \cdot \infty = \infty$

BUT: $\lim\limits_{x\to-\infty} x^2 e^x = \infty \cdot e^{-\infty} = \infty \times 0 \rightsquigarrow$ indeterminate

$= \lim\limits_{x\to-\infty} \dfrac{x^2}{e^{-x}} = \dfrac{\infty}{e^\infty} = \dfrac{\infty}{\infty} \rightsquigarrow$ l'Hôpital

$= \lim\limits_{x\to-\infty} \dfrac{2x}{-e^{-x}} = \dfrac{2(-\infty)}{-e^\infty} = \dfrac{\infty}{\infty} \rightsquigarrow$ l'Hôp

$= \lim\limits_{x\to-\infty} \dfrac{2}{e^{-x}} = \dfrac{2}{e^\infty} = 0 \Longrightarrow$ $y=0$ HA when $x \to -\infty$.

$f(x) = x^2 e^x \Rightarrow f'(x) = 2x e^x + x^2 e^x = \underbrace{e^x}_{\text{always } \oplus}(2x + x^2)$

$f'(x) = 0 \rightarrow e^x = 0$ NEVER

$2x + x^2 = 0 \Rightarrow x(2+x) = 0$ → $x = 0$, $x = -2$

x	-3	-2	-1	0	1
f'	$+$		$-$		$+$
f	↗		↘		↗

$f(0) = 0$

$f(-2) = (-2)^2 e^{-2} = \dfrac{4}{e^2} = \underbrace{\dfrac{4}{(2.71)^2}}_{\approx 0.5} < 1$

$f'(x) = e^x(x^2+2x)$

$f''(x) = e^x(x^2+2x) + e^x(2x+2) = e^x(x^2+4x+2)$ ⊕

$f''(x)=0 \Rightarrow x^2+4x+2=0 \Rightarrow x=-2\pm\sqrt{4-2} = -2\pm\sqrt{2}$

$\Rightarrow x \approx -2+1.4 = -0.6$

$x \approx -2-1.4 = -3.4$

x	$\overset{-4}{\uparrow}$	-3.4	$\overset{-1}{\uparrow}$	-0.6	$\overset{0}{\nearrow}$
f''	$+$	0	$-$	0	$+$
f	$\cup$	INF	$\cap$	INF	$\cup$

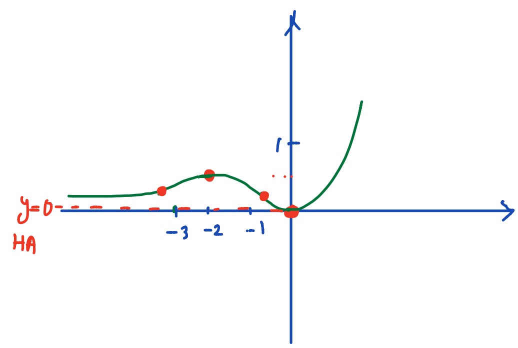
$f''(0) = ⊕(0+0+2) > 0$

$f''(-1) = ⊕(1-4+2) < 0$

$f''(-4) = ⊕(16-16+2) > 0$

Summary

x		-3.4		-2		-0.6		0	
f'	$+$ ↗		$+$ ↗		$-$ ↘		$-$ ↘		$+$ ↗
f''	$+$ ∪		$-$ ∩		$-$ ∩		$+$ ∪		$+$ ∪
f		INF		MAX		INF		Min	



$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

→ a constant

a) Domain = $\mathbb{R}$ +1 , Range = $y>0$, $(0,\infty)$ +1 → exp function is always ⊕.

b) $N'(x) = \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{-2(x-\mu)}{2\sigma^2}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right)$

$= \frac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} \; e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ → always ⊕

⊕ ←

$N'(x)$ → everywhere defined

↳ $N'(x) = 0$ → $(x-\mu)=0 \Rightarrow x=\mu$

↳ $e^{\bigcirc} = 0$ → NEVER

+2 pts

x	$\overset{\mu-1}{\uparrow}$	μ	test $\mu+1$ ↗
N'	$+$	0	$-$
N	↗	local MAX	↘

NO Local Min +1

This function is defined on $\mathbb{R}$ (NOT a closed interval) so we need to check the graph for global max/min.

c) $\mu = 0$, $\sigma = 1$ $\Rightarrow$ $N(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$N'(x) = \frac{-x}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \rightarrow N''(x) = \frac{1}{\sqrt{2\pi}}\left(-e^{-x^2/2} + (-x)\left(-\frac{2x}{2}\right)e^{\frac{-x^2}{2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(-e^{-x^2/2} + x^2 e^{-x^2/2}\right)$$

$$= 0$$

$$\underbrace{e^{-x^2/2}}_{\oplus}(-1 + x^2) = 0 \longrightarrow e^{\text{(shaded)}} = 0 \Rightarrow \text{NEVER}$$

$$\hookrightarrow x^2 = 1 \Rightarrow x = 1,\ x = -1$$

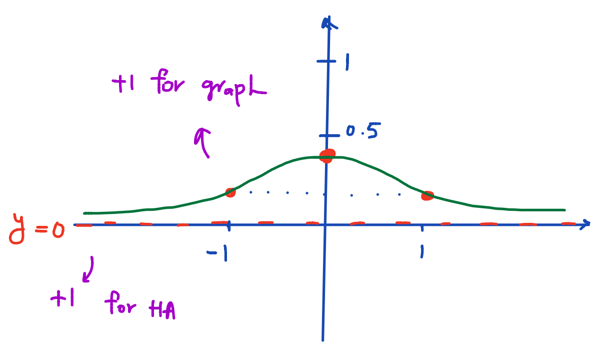
x	$\overset{-2}{\uparrow}$	-1	$\overset{0}{\uparrow}$	1	$\overset{2}{\uparrow}$
N''	$+$	0	$-$	0	$+$
N	$\cup$	INF	$\cap$	INF	$\cup$

$+2$

$+1$ { Global max at $x=0$
NO Global min

Summary

x		-1		0 "μ"		1	
N'	$+$		$+$	0	$-$		$-$
N''	$+$	0	$-$		$-$	0	$+$
N		INF		MAX		INF	



y-coordinates:

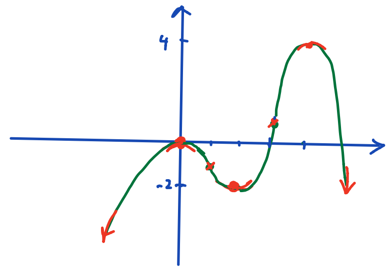
$$N(0) = \frac{1}{\sqrt{2\pi}} e^{0} = \frac{1}{\sqrt{2\pi}} \approx 0.4$$

$$N(1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} = N(-1) \approx 0.24$$

HA: $\lim\limits_{x\to\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = e^{-\infty} = 0$

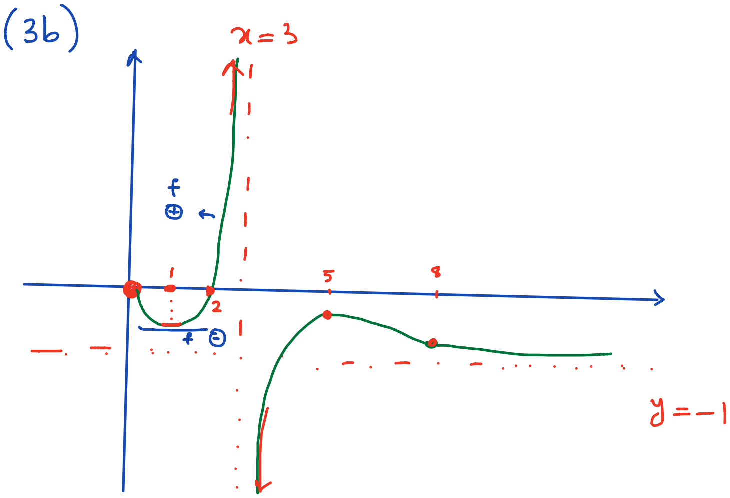
$\lim\limits_{x\to-\infty} N(x) = e^{-\infty} = 0$ $\Rightarrow y = 0$

(3a)



x		0		1		2		3		4	
f'	+	0	−		−	0	+		+	0	−
f''	−		−	0	+		+	0	−		−
f		MAX		INF		Min		INF		Max	

(3b)

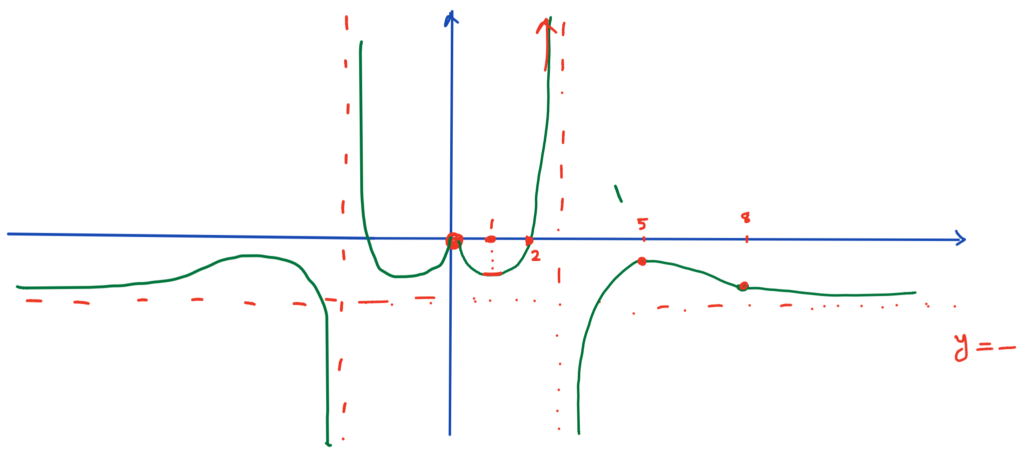


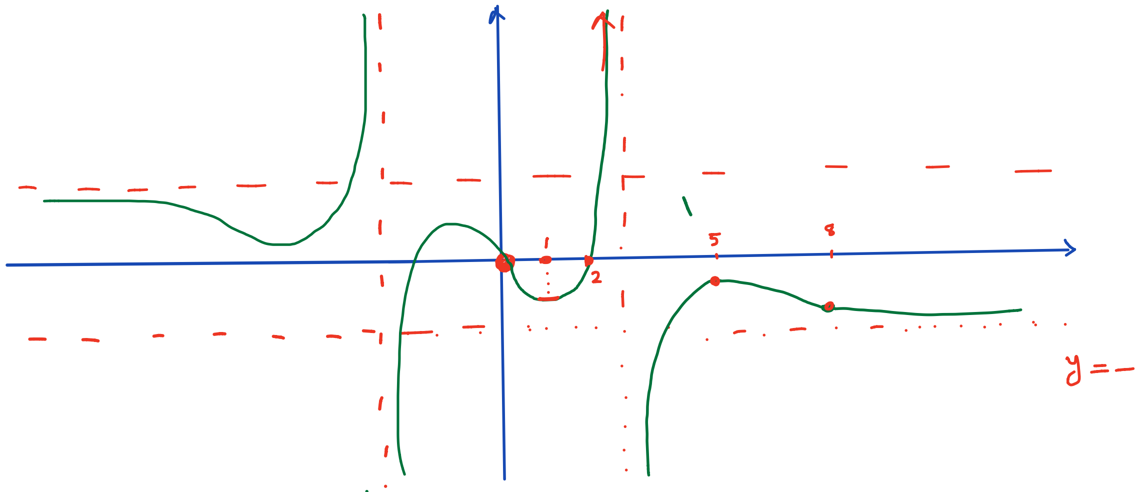
HA : $y = -1$

VA : $x = 3$

x	0		1		3		5		8	
f'		−	0	+		+	0	−		−
f''		+		+		−		−	0	+
f			Min		VA		Max		INF	

Bonus (i)





2
5
8
y = -1