Lecture 23, Mar 5 — Sketching a rational function

Example. Sketch $y = \dfrac{x^2+5x+4}{x^2}$.

f itself:

- Domain: $y = f(x)$ is undefined when $x^2 = 0 \Rightarrow x = 0$

  $\hookrightarrow$ $x \neq 0$

  possible VA

- Intercepts:

  $$\begin{cases} y\text{-int} \xrightarrow{x=0} \text{No } y\text{-int},\ f \text{ undefined at } x=0 \\ x\text{-int} \xrightarrow{y=0} \text{numerator} = 0 \end{cases}$$

  $\Rightarrow x^2+5x+4=0$

  $\Rightarrow (x+4)(x+1)=0$

  $\Rightarrow x=-4,\ x=-1$

  $\Rightarrow (-4,0),\ (-1,0)$

- Asymptotes:

VA: possible $x = 0$

Verify (important; limit definition) $\rightarrow$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x+4)(x+1)}{x^2} = \frac{4\times 1}{0^2} = \frac{4}{0^+} = +\infty$$

$$\lim_{x\to 0^-} f(x) = \frac{4\times 1}{0^2} = \frac{4}{0^+} = +\infty \Rightarrow x=0 \text{ VA}$$

HA : $\lim\limits_{x \to +\infty} \dfrac{x^2 + 5x + 4}{x^2} = \lim\limits_{x \to \infty} \dfrac{\cancel{x^2}\left(1 + \frac{5x}{x^2} + \frac{4}{x^4}\right)}{\cancel{x^2}}$

long way

$= \lim\limits_{x \to \infty} 1 + \underbrace{\cancel{\frac{5}{x}}}_{0} + \underbrace{\cancel{\frac{4}{x^2}}}_{0} = 1$

$\Rightarrow$ $y = 1$ is HA .

$\lim\limits_{x \to -\infty} \dfrac{x^2 + 5x + 4}{x^2} = \dfrac{\infty}{\infty} \longrightarrow$ l'Hôpital

Shorter way

$= \lim\limits_{x \to -\infty} \dfrac{2x + 5}{2x} = \dfrac{\infty}{\infty}$

$\Rightarrow y = 1$ HA

$= \lim\limits_{x \to -\infty} \dfrac{2}{2} = 1$

$\boxed{f'}$: $f(x) = \dfrac{x^2 + 5x + 4}{x^2}$

$\xrightarrow[\text{Rule}]{\text{Quotient}}$ $f'(x) = \dfrac{-5x - 8}{x^3}$

You compute it !

- f' is also undefined at $x = 0$
- $f'(x) = 0$ when $-5x - 8 = 0 \Rightarrow x = -\dfrac{8}{5}$

y-coordinate : $f\left(-\frac{8}{5}\right) = -\dfrac{9}{16} \rightsquigarrow \left(-\frac{8}{5}, -\frac{9}{16}\right)$

x	$-\infty$	$\overset{-2}{\uparrow}$	$-\frac{8}{5}$ $\overset{-1.\dots}{\nearrow}$	$\overset{-1}{\nearrow}$	0	$\overset{1}{\nearrow}$	$+\infty$
f'		$-$	0	$+$	∞	$-$	
f		$\searrow$	$\downarrow$ local min	$\nearrow$	$\downarrow$ VA f undefined at 0.	$\searrow$	

$f'(1) = \dfrac{-5-8}{1^3} < 0$

$f'(-1) = \dfrac{5-8}{(-1)^3} > 0$

$f'(-2) = \dfrac{10-8}{(-2)^3} < 0$

$\left(-\frac{8}{5}, -\frac{9}{16}\right)$ Min point

$\boxed{f''}$

$$f''(x) = \frac{10x+24}{x^4}$$

x^4 → always $\oplus$

f'' undefined at $x = 0$

$f''(x) = 0$ when $10x + 24 = 0 \Rightarrow x = -\frac{24}{10} = -\frac{12}{5}$

x	$-\infty$	$\overset{-3}{\uparrow}$	$-12/5$ $\overset{-2.\dots}{}$	$\overset{-2}{\uparrow}$	0	$\overset{1}{\nearrow}$	$+\infty$
f''		$-$	0	$+$	∞	$+$	
f		$\cap$	$\downarrow$ INF	$\cup$		$\cup$	

$f''(1) = \dfrac{10+24}{+} > 0$

$f''(-2) = \dfrac{-20+24}{+} > 0$

$f''(-3) = \dfrac{-30+24}{+} < 0$

y-coordinate:

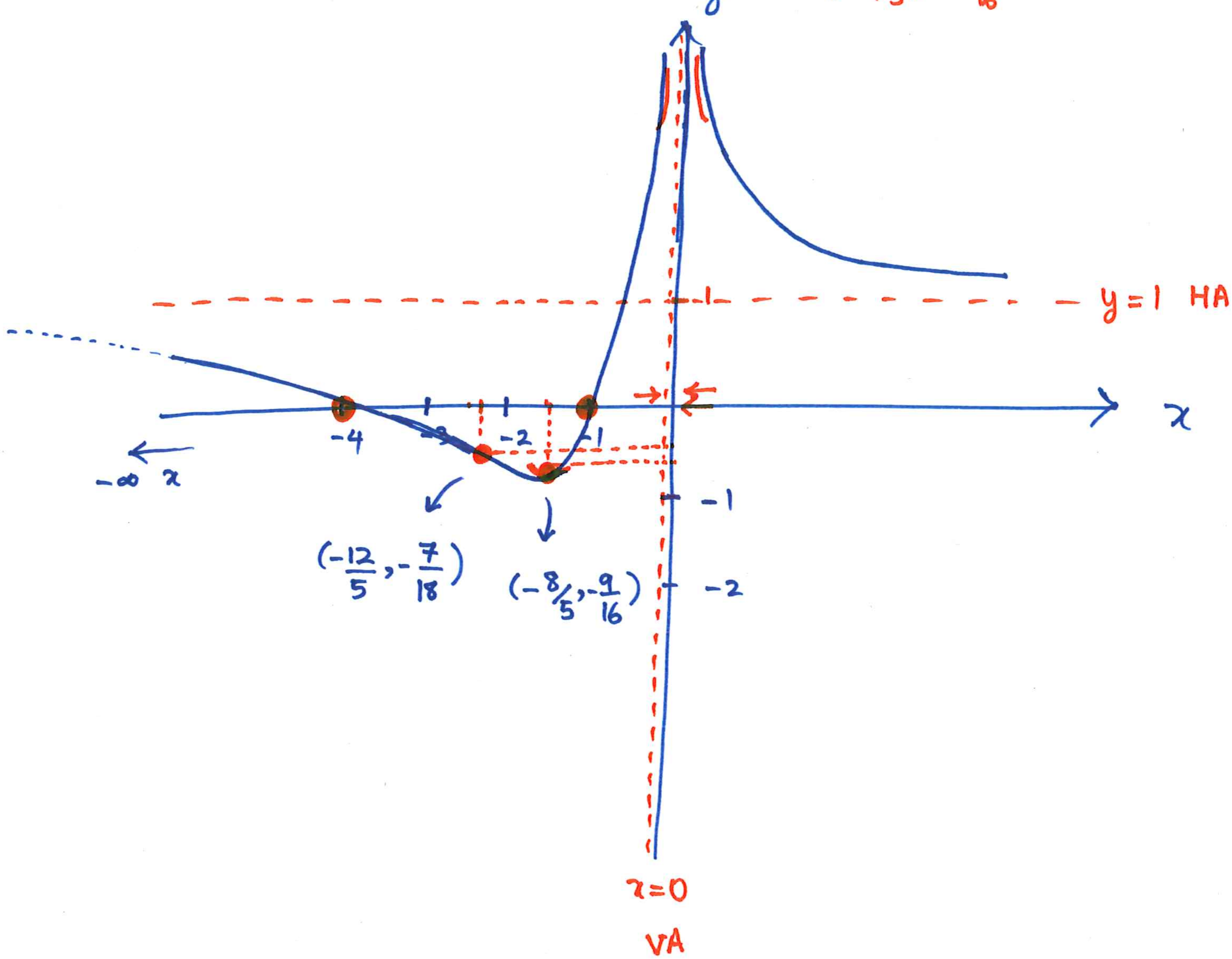
$f\left(-\frac{12}{5}\right) = -\frac{7}{18} \Rightarrow \left(-\frac{12}{5}, -\frac{7}{18}\right)$ INF point.

Summary Chart:

x	$-\infty$		$-12/5$		$-8/5$		0		∞
f'		$-$ ↘		$-$ ↘		$+$ ↗		$-$ ↘	
f''		$-$ ∩		$+$ ∪		$+$ ∪		$+$ ∪	
f			INF		Min		VA		

INF $\left(-\frac{12}{5}, -\frac{7}{18}\right)$

Min $\left(-8/5, -\frac{9}{16}\right)$



Example 2. Sketch $y = \dfrac{x^2}{1-x^2}$

given that $y' = \dfrac{2x}{(1-x^2)^2}$

and $y'' = \dfrac{2+6x^2}{(1-x^2)^3}$.

- **Domain:** $1 - x^2 = 0 \Rightarrow x^2 = 1$

$\Rightarrow \underline{x=1}\ ,\ \underline{x=-1}$

possible VA

Domain: everywhere except $x = 1, -1$

- **Intercepts:**

$$\begin{cases} y\text{-int} \xrightarrow{x=0} f(0) = \dfrac{0^2}{1-0^2} = 0 \Rightarrow (0,0) \\ x\text{-int} \xrightarrow{y=0} \text{top} = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0 \end{cases}$$

- **Asymptotes:**

VA : $\displaystyle\lim_{\substack{x \to 1^+ \\ \downarrow \\ 1.1}} \frac{x^2}{\underset{\substack{\downarrow \\ 1-1.1=-0.1}}{(1-x)}(1+x)} = \frac{1^2}{(1-1)\,2} = \frac{1}{0^-} = -\infty$

$\Rightarrow x = 1$ is VA.

$\displaystyle\lim_{\substack{x \to 1^- \\ \downarrow \\ 0.9}} \frac{+^2}{2 \cdot \underset{\substack{\downarrow \\ 1-0.9=0^+}}{(1-1^-)}} = +\infty$

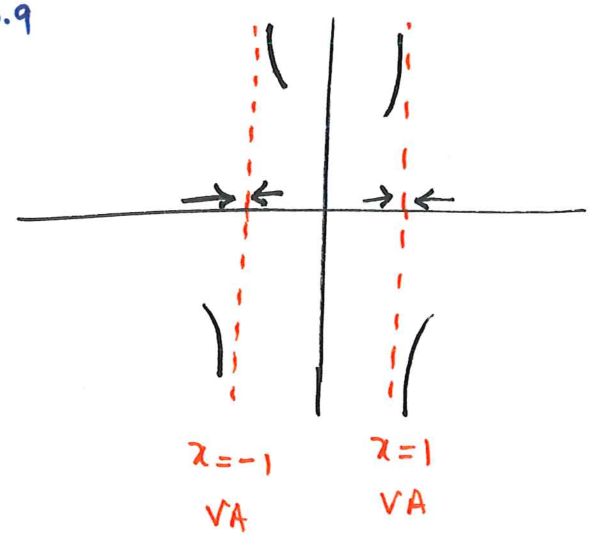
VA : $\lim\limits_{x \to -1^-} \dfrac{x^2}{(1-x)(1+x)} = \dfrac{(-1)^2}{(1-(-1))(1+(-1))} = \dfrac{1}{2 \times 0^-} = -\infty$

$x \to -1^- \to -1.1$ $\qquad 1 - 1.1 = -0.1$

$\Rightarrow x = -1$

$\lim\limits_{x \to -1^+} \dfrac{(-1)^2}{(1-(-1))(1+(-1^+))} = +\infty$

$x \to -1^+ \to -0.9$ $\qquad 1 - 0.9 = 0^+$



Rough Sketch of f around VA.