March 14
Lecture 27

Optimization: Minimizing or Maximizing a quantity.

Find global max/min.

Let's see two examples:

Example 1. We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all the material is used in the construction process, determine the maximum volume that the box can have.

constraint.

optimized quantity.

Example 2. A rectangle is inscribed with its upper vertices on the parabola $y = -x^2 + 3$ and its lower vertices on the x-axis. What are the dimensions of such a rectangle with the greatest possible area?

constraint

optimized quantity.

To solve an optimization problem we need to follow several steps. The first few steps are to translate the words into formulas and then we use some algebra and Calculus for the computational parts.

→ Steps to solve a typical optimization problem

1) Read the question carefully and look for a quantity that should be optimized.

   ↓ words such as: max, min, largest, smallest longest, shortest, closest ...

2) Draw a diagram and visualize the given scenario.

3) Make a formula for the optimized quantity.

   ↓ We need to label the diagram.

4) Check the extra info given and find a formula for that. → Constraint.

TRANSLATING WORDS

5) Go back to the optimized quantity and make that a function of only one variable.

   ↓ Use constraint and some algebra.

   Also find the bounds for the variables.

ALGEBRA

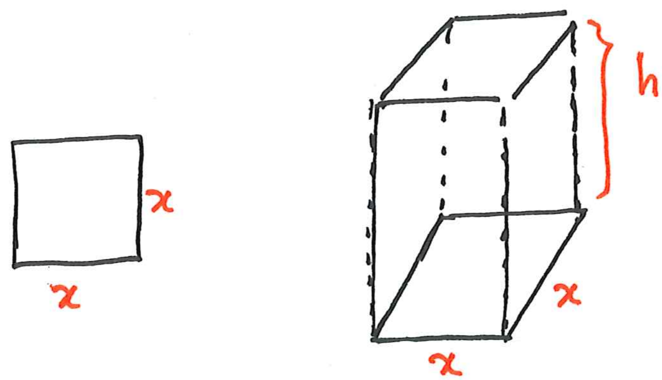
6) Use Calculus to maximize/minimize the optimized quantity.

→ CALCULUS :)

Ex 1.

1) Optimized quantity = Volume of the box.

2)



3) Volume = length · width · height

$$V = x \cdot x \cdot h$$

$$\boxed{V = x^2 h} \longrightarrow \text{Optimized quantity}$$

4) Extra info :

Surface area of the box = $10 \ m^2$

$$2\,\text{Area}_{(\text{bottom})} + 4\,A_{(\text{side})} = 10$$

$$\boxed{2x^2 + 4xh = 10} \longrightarrow \text{Constraint}$$

5) It is easier to isolate h and solve this equation

for h : $4xh = 10 - 2x^2$

divide by 2

$$2xh = 5 - x^2$$

$$\Rightarrow \boxed{h = \frac{5 - x^2}{\;}} \longrightarrow h \text{ in terms of } x$$

Rewrite the optimized quantity:

$$V = x^2 h = \cancel{x^2} \cdot \frac{5 - x^2}{2\cancel{x}}$$

V is a function of one variable now.

$$\boxed{V(x) = \frac{5x - x^3}{2} \qquad x > 0}$$

6) Find Critical Numbers & local extrema.

$$V'(x) = \frac{1}{2}\left(5 - 3x^2\right)$$

→ everywhere defined

→ $= 0$

$$5 - 3x^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{5}{3} \quad \Rightarrow \quad \boxed{x = \sqrt{\frac{5}{3}}}$$

$$x = -\sqrt{\frac{5}{3}}$$

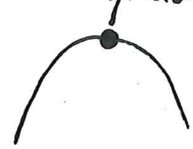
We need to check if

$x = \sqrt{\frac{5}{3}}$ gives the local max.

apply the 2^{nd} derivative test: → Next page

$$V''(x) = \frac{1}{2}(-6x) \xrightarrow{x = \sqrt{\frac{5}{3}}} V''\left(\sqrt{\frac{5}{3}}\right) = \frac{1}{2}\left(-6\sqrt{\frac{5}{3}}\right) < 0$$

max



$x = \sqrt{\frac{5}{3}}$ gives a local max

≈ 1.29

To verify if a critical number is a local max or a local min, we have two options.

1. First Derivative test $\rightsquigarrow$ Sign Chart for f'

f' $\oplus$ $\nearrow$ $\ominus$ $\searrow$

local max

f' $\ominus$ $\searrow$ $\nearrow$ $\oplus$

local min

2. Second derivative Test $\rightsquigarrow$ NO sign chart needed.

Check the sign of f'' at the critical number.

If "c" is a critical number of function f such that $f'(c) = 0$

If $f''(c) > 0 \Longrightarrow x = c$ gives a local min

If $f''(c) < 0 \Rightarrow x = c$ gives a local max.

If $f''(c) = 0 \Rightarrow$ This test does NOT work.

Ex 1 continued

Exercise. a) Find the volume of this box when $x = \sqrt{\frac{5}{3}} \approx 1.29$

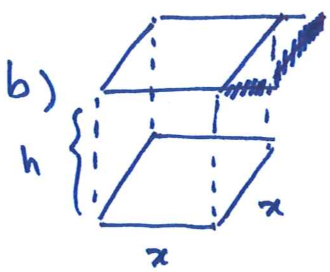
b) Find the dimensions of this box $\rightsquigarrow$ height?

$$V(x) = \frac{1}{2}(5x - x^3) \qquad x > 0$$

a) $x = \frac{\sqrt5}{\sqrt3} \Rightarrow V\left(\frac{\sqrt5}{\sqrt3}\right) = \frac{1}{2}\left(\frac{5\sqrt5}{\sqrt3} - \sqrt{\left(\frac{5}{3}\right)^3}\right) \approx 2.15 \text{ m}^3$

This is the max volume.

b)



Let's find h:

We know $h = \frac{1}{2}\left(\frac{5 - x^2}{x}\right)$

$$\xrightarrow{x = \frac{\sqrt5}{\sqrt3}} h = \frac{1}{2}\left(\frac{5 - \frac{5}{3}}{\frac{\sqrt5}{\sqrt3}}\right) = \frac{1}{2}\left(\frac{\frac{10}{3}}{\frac{\sqrt5}{\sqrt3}}\right)$$

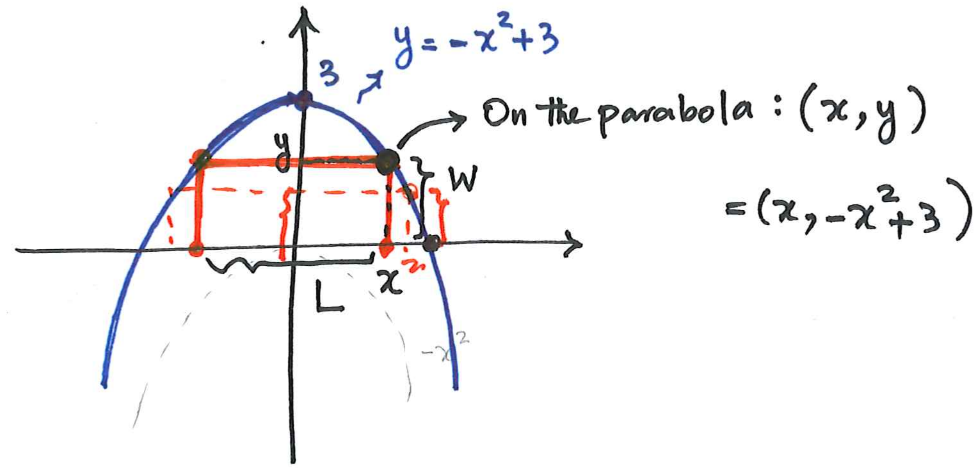
$$= \frac{1}{\cancel{2}}\,\frac{\overset{\sqrt5\ \cancel{5}}{\cancel{10}} \cdot \cancel{\sqrt3}}{\cancel{\sqrt5} \cdot \underset{\sqrt3}{\cancel{3}}} = \frac{\sqrt5}{\sqrt3}$$

We see that $x = \frac{\sqrt5}{\sqrt3}$ and $h = \frac{\sqrt5}{\sqrt3}$ $\Longrightarrow$ maximum volume is when all sides of the cube are equal $\rightsquigarrow$ perfect square cube!

Ex 2.

1. Optimized quantity = Area of a rectangle.

2. Diagram.



3. Area = length · width

$$\boxed{A = W \cdot L} \longrightarrow \text{Optimized quantity.}$$

4. Constraint : parabola : $y=-x^2+3$.

5. What is the relation between W, L, x and y?

$$\left.\begin{aligned} W &= y \\ L &= 2x \end{aligned}\right\} \Longrightarrow A = y \cdot 2x$$

$$\boxed{\begin{aligned} A(x) &= (-x^2+3)\cdot 2x \\ 0 \leq\ &x \leq \sqrt{3} \end{aligned}}$$

$0 \leq x \leq \underbrace{x\text{-int}} \Longrightarrow 0 \leq x \leq \sqrt{3}$

$y=-x^2+3=0 \Rightarrow -x^2=-3 \Rightarrow x^2=3 \Rightarrow \boxed{x=\sqrt{3}}$

$x=-\sqrt{3}$

we Finish it next class.