March 14

Lecture 27

Optimization : Minimizing or Maximizing a quantity.

Find global max / min.

Let's see two examples:

Example 1 . We want to construct a box with a square base and we only have $\boxed{10\ m^2 \text{ of material}}$ to use in construction of the box. Assuming that all the material is used in the construction process, determine $\boxed{\text{the maximum volume}}$ that the box can have.

constraint.

optimized quantity.

Example 2 . A rectangle is inscribed with its upper vertices on the parabola $\boxed{y = -x^2 + 3}$ and its lower vertices on the x-axis. What are the dimensions of such a rectangle with $\boxed{\text{the greatest possible area}}$?

constraint

optimized quantity.

To solve an optimization problem we need to follow several steps. The first few steps are to translate the words into formulas and then we use some algebra and Calculus for the computational parts.

Ex 1. Continued (Example of making a box with a square base.)

Check whether $x = \sqrt{\frac{5}{3}}$ gives a global max.

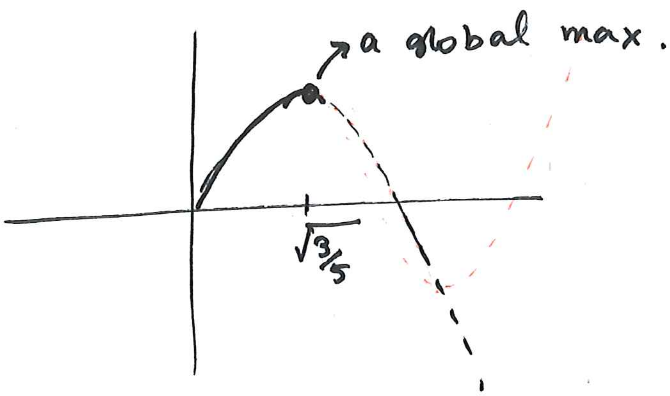
$$V(x) = \frac{5x - x^3}{2} \qquad x > 0 \qquad (0, \infty)$$

$$V'(x) = \frac{1}{2}(5 - 3x^2)$$

We did part of this example in last class.

$V''(x) = \frac{1}{2}(-6x) \rightsquigarrow$ always negative $\rightarrow$ always concave down.

Method: Roughly visualize the graph of the function.



If the function continuous above the local max then it should have a local min as well, but this is impossible.

$\Longrightarrow$ $x = \sqrt{\frac{5}{3}}$ gives a global max.

Ex 1 continued

Exercise. a) Find the volume of this box

when $x = \sqrt{\frac{5}{3}} \approx 1.29$

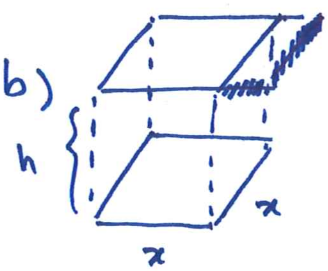
b) Find the dimensions of this box ⇝ height?

$$V(x) = \frac{1}{2}(5x - x^3) \qquad x > 0$$

a) $x = \frac{\sqrt{5}}{\sqrt{3}} \Rightarrow V\left(\frac{\sqrt{5}}{\sqrt{3}}\right) = \frac{1}{2}\left(\frac{5\sqrt{5}}{\sqrt{3}} - \sqrt{\left(\frac{5}{3}\right)^3}\right) \approx 2.15 \text{ m}^3$

This is the max volume.

b)



Let's find h:

We know $h = \frac{1}{2}\left(\frac{5 - x^2}{x}\right)$

$\xrightarrow{x = \frac{\sqrt{5}}{\sqrt{3}}} \quad h = \frac{1}{2}\left(\frac{5 - \frac{5}{3}}{\frac{\sqrt{5}}{\sqrt{3}}}\right) = \frac{1}{2}\left(\frac{\frac{10}{3}}{\frac{\sqrt{5}}{\sqrt{3}}}\right)$

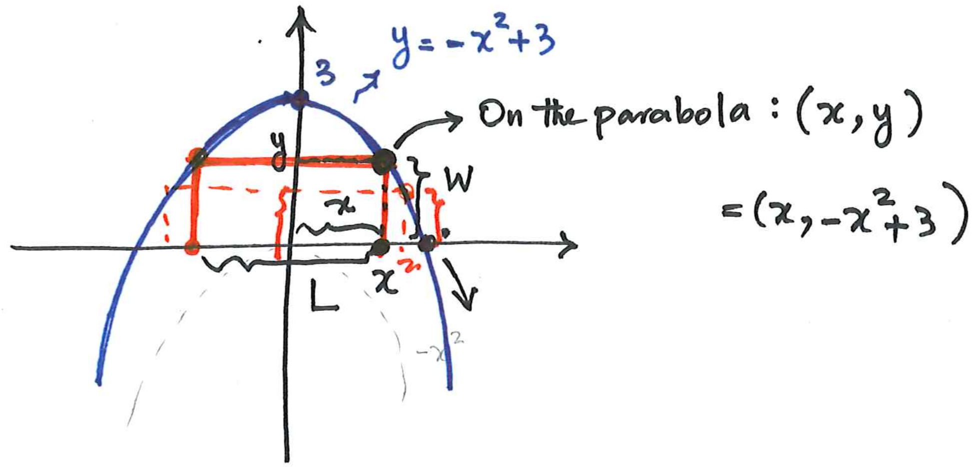
$$= \frac{1}{\cancel{2}} \; \frac{\cancel{10}^{\sqrt{5}\,\cancel{5}} \cdot \cancel{\sqrt{3}}}{\cancel{\sqrt{5}} \cdot \cancel{3}_{\sqrt{3}}} = \frac{\sqrt{5}}{\sqrt{3}}$$

We see that $x = \frac{\sqrt{5}}{\sqrt{3}}$ and $h = \frac{\sqrt{5}}{\sqrt{3}}$ $\Rightarrow$ maximum volume is when all sides of the cube are equal ⇝ perfect square cube!

Ex 2. Rectangle inside a parabola:

1. Optimized quantity = Area of a rectangle.

2. Diagram.



3. Area = length · width

$$\boxed{A = W \cdot L} \longrightarrow \text{Optimized quantity.}$$

4. $\boxed{\text{Constraint}}$: parabola : $y = -x^2+3$.

5. What is the relation between W, L, x and y?

$$\left.\begin{aligned} W &= y \\ L &= 2x \end{aligned}\right\} \Longrightarrow A = y \cdot 2x$$

$$A(x) = (-x^2+3) \cdot 2x$$

$0 \leqslant x \leqslant x\text{-int} \Longrightarrow 0 \leqslant x \leqslant \sqrt{3}$

$y = -x^2+3 = 0 \Rightarrow -x^2 = -3 \Rightarrow x^2 = 3 \Rightarrow \boxed{x = \sqrt{3}}$

$x = -\sqrt{3}$

~~we finish it next class~~.

5. $A(x) = (-x^2+3)\cdot 2x = -2x^3 + 6x$

6. $0 \leq x \leq \sqrt{3}$ → **closed interval**

7. $A'(x) = -6x^2 + 6 = 0$

$\Rightarrow -6x^2 = -6 \Rightarrow x^2 = 1$

$\Rightarrow \boxed{x = 1}, \quad x = -1$

Option 1: Sign chart for A'

1st derivative test ?

Option 2. 2nd derivative test :

$A''(x) = -12x$

$A''(1) = -12 < 0$

testing endpoints: Is $x=1$ a global max ?

$A(0)$, $A(\sqrt{3})$, $A(1)$

$A(x) = 2x(-x^2+3)$

$A(0) = 0$, $A(\sqrt{3}) = 2\sqrt{3}\,(\underbrace{\cancel{-3+3}}_{0}) = 0$

$A(1) = 2(-1+3) = 2.2 = 4$ → largest

The area function has a global max at $x=1$.

The value of the global max is $\boxed{A(1)=4}$

The Question asks for the dimension:

$$L = 2x \xrightarrow{x=1} \boxed{L=2}$$

$$W = y \xrightarrow[x=1]{y=-x^2+3} y=-1+3=2 \Rightarrow \boxed{W=2}$$

$\Rightarrow$ Max area is when the rectangle is a square.