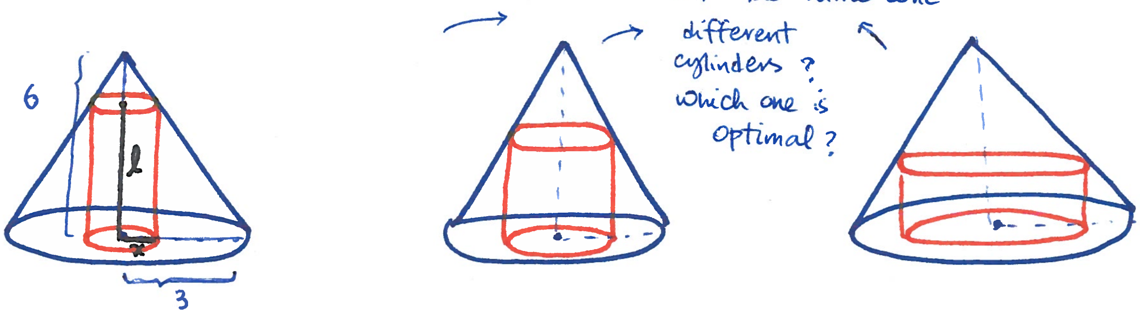
March 23 (Lecture 31)

(Q9) Worksheet. Optimized function : Volume of the cylinder inscribed in the cone.

(a)

Different situations may happen:



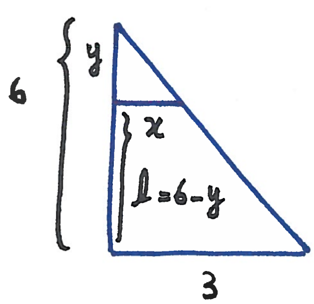
We have a fixed cone, we'd like to choose the cylinder with maximum volume.

Formulate Objective function : $V = \pi x^2 \cdot l$

radius of cylinder base (x); height of the cylinder (l).

Constraint : A given cone with fixed dimension.

How to reduce V to a function of one variable? Similar triangles.



$$\frac{x}{3} = \frac{y}{6} \Rightarrow 3y = 6x \Rightarrow y = 2x$$

So : $\boxed{l = 6 - y = 6 - 2x}$

Rewrite V : $V = \pi x^2 \cdot l = \pi x^2 \cdot (6 - 2x)$

$$\Rightarrow \boxed{V(x) = 6\pi x^2 - 2\pi x^3}$$

Domain : $\boxed{0 \leq x \leq 3}$

$$V(x) = 6\pi x^2 - 2\pi x^3 \quad , \quad 0 \le x \le 3$$

Calculus: Maximize V :

$$V'(x) = 6\pi(2x) - 2\pi(3x^2)$$

$$\Rightarrow V'(x) = 12\pi x - 6\pi x^2 = 0$$

factor

$$\Rightarrow 6\pi x(2 - x) = 0 \quad \rightarrow \boxed{x = 0} \text{ endpoint}$$

$$\rightarrow 2 - x = 0 \Rightarrow \boxed{x = 2}$$

- Check $V''(x)$ at $x = 2$?

$$V''(x) = 12\pi - 12\pi x \xrightarrow{x=2} V''(2) = 12\pi - 24\pi = -12\pi < 0$$

- Check global max:

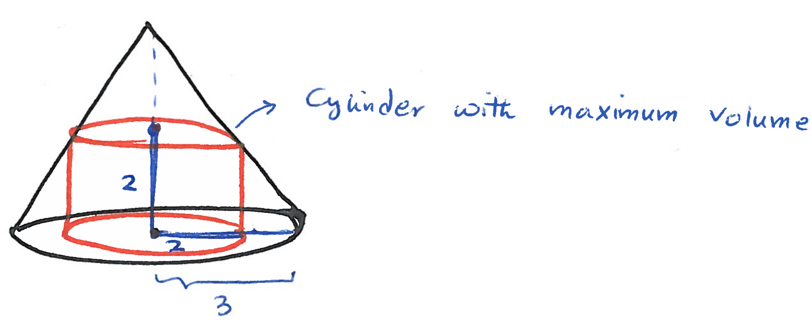
$$x = 2 \longrightarrow V(2) = 6\pi(2)^2 - 2\pi(2)^3 = 24\pi - 16\pi = 8\pi \rightsquigarrow \text{largest volume}$$

$$x = 0 \longrightarrow V(0) = 0$$

$$x = 3 \longrightarrow V(3) = 6\pi(3)^2 - 2\pi(3)^3 = 54\pi - 54\pi = 0$$

So the maximum volume is when the cylinder has radius $x = 2$ and height $\ell = 6 - 2x = 6 - 4 = 2$

$\Rightarrow$ Equal radius and height



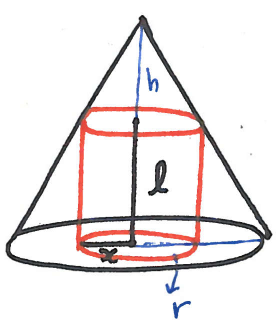
(b) In part (b) the dimension of the cone is not given as a number, but as parameters r and h.

a quantity that can take any (permissible) value but it is viewed as a constant.

Important Note: A parameter is different from a variable, although they're both denoted by letters. Variable is the input for a function, ~~variable~~ Parameter is a fixed quantity.

For example, recall that we denote a general quadratic function by

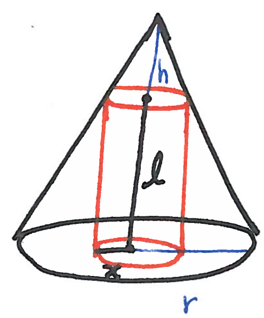
$$y = ax^2 + bx + c \rightsquigarrow \begin{array}{l} x : \text{variable} \\ a, b, c : \text{parameters.} \end{array}$$



h, r : fixed

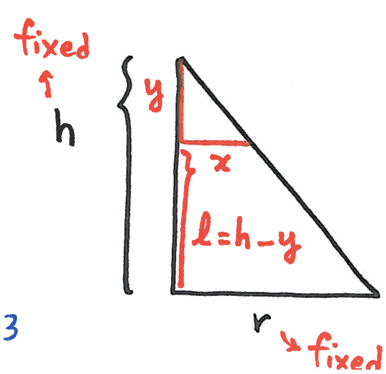
x, ℓ : changing

→ Same cone different cylinders ←



Optimized function: Volume of Cylinder : $V = \pi x^2 \cdot \ell$

Constraint: Cone with radius r, height h



Similar Triangles : $\dfrac{x}{r} = \dfrac{y}{h}$ (x: variable, r: fixed, y: variable, h: fixed)

$$\xrightarrow[\text{multiply}]{\text{cross}} xh = ry \xrightarrow[\text{by } r]{\text{divide}} \frac{xh}{r} = y$$

So $\ell = h - y = h - \frac{xh}{r}$ fixed, variable, fixed

Rewrite the Volume : $V = \pi x^2.$

$$V = \pi x^2 \left(h - \frac{xh}{r} \right)$$

So $V(x) = \underbrace{\pi h}_{\text{Constant}} x^2 - \underbrace{\frac{\pi h}{r}}_{\text{Constant}} x^3$

Domain : $0 \leq x \leq r$

Calculus :

$$V'(x) = \pi h (2x) - \frac{\pi h}{r}(3x^2)$$

$$\xrightarrow{\text{factor}} \quad V'(x) = \pi h x \left(2 - \frac{3x}{r}\right)$$

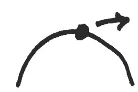
$$\xrightarrow{V'(x)=0} \quad \pi h x = 0 \longrightarrow \boxed{x = 0} \text{ endpoint}$$

$$2 - \frac{3x}{r} = 0 \xrightarrow[\text{Solve for } x]{} 2 = \frac{3x}{r} \Rightarrow 2r = 3x \Rightarrow \boxed{\frac{2r}{3} = x}$$

- Is $x = \frac{2r}{3}$ a local max ?

$$V''(x) = \pi h \, 2 - \frac{\pi h}{r} 6x \xrightarrow{x = \frac{2r}{3}} V''\left(\frac{2r}{3}\right) = 2\pi h - \frac{\pi h}{\cancel{r}} \cdot \cancel{6}^{\,2} \cdot \frac{2\cancel{r}}{\cancel{3}}$$

$$= 2\pi h - 4\pi h = -2\pi h < 0$$


local max

- Is it a global max : Compare values

$V(0) = 0$

$V(r) = \pi h r^2 - \frac{\pi h}{r} r^3 = 0$

$$V\left(\frac{2r}{3}\right) = \pi h \left(\frac{2r}{3}\right)^2 - \frac{\pi h}{r}\left(\frac{2r}{3}\right)^3 = \pi h \frac{4r^2}{9} - \pi h \frac{8r^2}{27} = \pi h \left(\frac{12r^2 - 8r^2}{27}\right)$$

$\Rightarrow V\left(\frac{2r}{3}\right) = \pi h \left(\frac{4r^2}{27}\right)$ ⇝ largest volume.

So the largest volume is when the radius of the cylinder is $x = \frac{2r}{3}$ and its height is $l = h - \frac{xh}{r} = h - \frac{2r}{3} \cdot \frac{h}{r}$

so $l = h - \frac{2h}{3} = \frac{h}{3}$, the maximum volume is

$$V\left(\frac{2r}{3}\right) = \frac{4\pi h r^2}{27}$$

- Note: If you plug in $h = 6$ and $r = 3$, you'll get exactly the same values as in part (a).