l'Hôpital's Rule (other indeterminate limits)

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lecture 21

Last class, we learned that l'Hôpital's rule can be used to find the limits which are in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

For example; $\lim\limits_{x \to 0} \frac{x}{\tan x} = \frac{0}{\tan 0} = \frac{0}{0}$!

l'Hôpital $= \lim\limits_{x \to 0} \frac{1}{1+\tan^2 x} = \frac{1}{1+\tan^2 0} = 1$

$\circledast$ $1+\tan^2 x = \sec^2 x$

and $\lim\limits_{x \to \infty} \frac{2+\ln x}{x^2+7} = \frac{2+\ln(\infty)}{\infty+7} = \frac{\infty}{\infty}$!

l'Hôpital $= \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{2x} = \lim\limits_{x \to \infty} \frac{1}{2x^2} = 0$

* $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \times d}{b \times c}$

Today, we use l'Hôpital's rule to deal with other indeterminate forms, these other forms are:

$\underbrace{0 \times \infty}$ → $\lim\limits_{x \to 0^+} x \ln x$

$\underbrace{\infty - \infty}$ → $\lim\limits_{x \to 0} \frac{1}{x} - \frac{1}{x^2}$

$\underbrace{1^\infty}$ → $\lim\limits_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

$\underbrace{0^0}$ → $\lim\limits_{x \to 0} x^x$

$\underbrace{\infty^0}$ → $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

In this course, we cover $0 \times \infty$ and $\infty - \infty$ cases.

In these other indeterminate forms, we need to manipulate the function and transform it into a quotient form $\frac{f(x)}{g(x)}$ so that l'Hôpital's rule can be applied.

Example 1. $\lim_{x \to \infty} x^2 e^{-x} = \infty \cdot e^{-\infty} = \infty \times 0$

* Recall: $\lim_{x \to \infty} e^{-x} = 0$ or $\lim_{x \to -\infty} e^{x} = 0$

$\downarrow$

Write $x^2 e^{-x}$ in a quotient form.

$\downarrow$

There's a negative exponent which can become the denominator of a fraction:

$$x^2 e^{-x} = \frac{x^2}{e^x}$$

So $\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = \frac{\infty}{e^{\infty}} = \frac{\infty}{\infty}$!

$\downarrow$

Now apply l'Hôpital's rule

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$$

once again l'Hôpital

$$= \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0 \checkmark$$

One trick: one way to write multiplication of two functions as a division is to look for negative exponents and bring them down to the denominator and make a fraction.

$$x^{-2} e^{x} = \frac{e^x}{x^2}, \quad (\ln x) e^{-x} = \frac{\ln x}{e^x}$$

Example 2.

$$\lim_{x \to 0^+} x \ln x = 0 \ln(0) = 0 \times \infty$$

WHY just 0+?

⊛ In this example, there's NO negative exponent so how can we write $x \ln x$ in the form of a fraction:

General trick for $0 \times \infty$ cases:

two terms are multiplied "ab" we can use an algebra trick to transform it to a division.

$$a \cdot b = \frac{a}{\frac{1}{b}} = \frac{b}{\frac{1}{a}}$$

Keep one and divide it by the reciprocal of the other.

Now let's find $\lim_{x \to 0^+} x \ln x$.

Rewrite: $x \ln x = \dfrac{\ln x}{\frac{1}{x}}$

So
$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

apply l'Hôpital

⊛ $$\frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\frac{x^2 \times 1}{x \times 1} = -x$$

$$= \lim_{x \to 0^+} -\frac{x^{\cancel{2}}}{\cancel{x}} = \lim_{x \to 0^+} -x = 0 \checkmark$$

Example 3.

$$\lim_{x \to 0} \frac{1}{x} - \frac{1}{x^2} = \frac{1}{0} - \frac{1}{0} = \infty - \infty \rightsquigarrow \text{indeterminate}$$

$(\infty - \infty \neq 0)$

Trick: common denominator

$$= \lim_{x \to 0} \frac{x - 1}{x^2} = \frac{0 - 1}{0^2} = \frac{-1}{0^+} = -\infty$$

NO l'Hôpital is required. We did NOT get $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

An important final note.

l'Hôpital's Rule is NOT the quotient rule.

Compare : quotient rule : $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'g - fg'}{g^2}$

But in l'Hôpital's rule we are just replacing top and bottom by their derivatives. It is NOT a differentiation rule $\rightsquigarrow$ $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Practice Problems. Find the following limits, use l'Hôpital's rule if needed.

a) $\lim_{x\to 0} \frac{\ln(1+3x)}{5x}$

b) $\lim_{x\to\infty} (x+1)^3 - x^3$

c) $\lim_{x\to\infty} e^{-x} + x^{-1}$

d) $\lim_{x\to 1} \frac{2}{x-1} - \frac{1}{x^2-1}$

e) $\lim_{x\to 0^+} \sqrt{x}\,\ln(x)$

f) $\lim_{x\to\infty} x^3 e^{-3x^2}$

g) $\lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right)$

h) $\lim_{x\to\infty} x^2 e^{-x}$

i) $\lim_{x\to 2^+} e^{\frac{1}{2-x}}$

j) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x}$