March 19

Complete worksheet from last week, and one example from the new worksheet.

Translate words into formulas.

In the following, an optimization problem is given. We would like to relate the phrases given in the problem statement to the steps of the optimization strategy.

- Find the dimensions of a $\boxed{\text{rectangle with area 36 ft}^2}$ whose perimeter is as small as possible.

(a) With which step of the strategy does the boxed piece match and why?

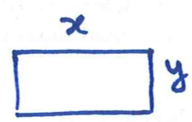
Step 4 $\rightsquigarrow$ This is the constraint because we are given a fixed value for area.

- Find the dimensions of a rectangle with area 36 ft^2 whose $\boxed{\text{perimeter is as small as possible}}$.

(b) With which step of the strategy does the boxed piece match and why?

This is the objective (optimized) function $\rightsquigarrow$ Step 1

(c) *Objective function*: Write a formula that represents the quantity to be optimized.



objective function: perimeter of a rectangle

$$P = 2x + 2y$$

(d) *Constraint*: Write an auxiliary formula given by the constraint and use that to write the objective function in terms of one variable.

Constraint : a fixed area for the rectangle.

$$xy = 36$$

(e) Do the rest of the steps and solve the problem completely.

Step 5 : By constraint : $y = \frac{36}{x}$

So rewrite the objective function:

$$P = 2x + 2y = 2x + 2 \cdot \frac{36}{x}$$

$$\Rightarrow P(x) = 2\left(x + \frac{36}{x}\right) \quad \text{and} \quad x > 0 \longrightarrow \text{step 6}$$

Step 7 : $P'(x) = 2\left(1 - \frac{36}{x^2}\right) \longrightarrow$ NOT defined at $x=0 \rightsquigarrow$ NOT in the domain

$$P'(x) = 0 \Rightarrow \frac{x^2 - 36}{x^2} = 0 \Rightarrow x^2 - 36 = 0 \Rightarrow \boxed{x = 6},\ x = -6$$

in the domain

Final Steps :

- Check if $x=6$ is the minimum

  $\downarrow$

  2nd derivative test : $P'(x) = 2(1 - 36x^{-2})$

$$\Rightarrow P''(x) = 2(2 \cdot 36\, x^{-3}) = 4 \cdot 36 \cdot \frac{1}{x^3}$$

$$\Rightarrow P''(6) > 0 \quad \to \text{Min}$$

So we are sure that $x=6$ gives a local min.

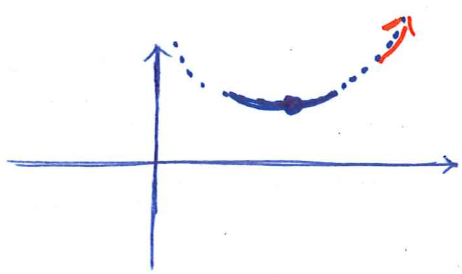
- Is $x=6$ a global min ?

Domain: open interval : $(0, \infty)$

$\downarrow$ Visualize the graph: $P(x) = 2\left(\frac{x^2+36}{x}\right)$

Note that $\lim_{x \to \infty} \frac{2(x^2+36)}{x} = \frac{\infty}{\infty} \xrightarrow{\text{l'Hôp}} = \lim_{x\to\infty} \frac{2 \cdot 2x}{1} = \infty$

So the function grows upward and it has a local min at $x=6$



$\rightarrow$ if there's any other minimum, then the function must have a local max too, which is impossible

NOT possible

$P(6) = 2\left(6 + \frac{36}{6}\right) = 2(6+6) =$

- Go back and re-read the question:

? Dimension ? $\rightsquigarrow x = 6$

$\rightsquigarrow y = \frac{36}{x} = \frac{36}{6} = 6$ $\Longrightarrow$ minimum perimeter is when the rectangle becomes a square.

Worksheet : Some Optimization Problems.

1. **Lifeguard Problem.**

You are a lifeguard at a beach in Vancouver. One day, as you are sitting on your lifeguard chair, you see a swimmer calling for help. The swimmer appears to be roughly 120 feet out to sea (on a straight line between the swimmer and the shore), and the lifeguard station is roughly 300 feet down the beach from the nearest point on shore to the swimmer. You can run at 13 feet per second along the beach, and you can swim at 5 feet per second. At what point along the shoreline should you enter the water so as to reach the swimmer as quickly as possible? How far do you run and how far do you swim? → Constraint

Minimize time ←

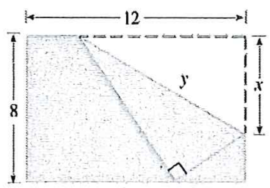
2. We want to make a cylindrical can with volume 128 in^3 by cutting its top and bottom from squares of metal and forming its curved side by bending a rectangular sheet of metal to match its ends. What radius r and height h of the can will minimize the total amount of material required for the rectangle and the two squares?

   How would it be different if it is an open top can?

3. A 2 feet piece of wire is cut into two pieces and one piece is bent into a square and the other is bent into a circle. Where should the wire be cut so that the total area enclosed by both is maximum?

4. You want to make a box with an open top such that the width of its base is one fourth of its length. If the material to build the bottom is 5\$ per square inch and the material for the sides costs 20\$ per square inch. What are the dimensions of the cheapest box which will hold 100 cubic inches of water?

5. The upper right-hand corner of a piece of paper, 12 in. by 8 in. as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



6. Find the points on the parabola $y = x^2 + 1$ that are closest to $(0, 2)$.

7. Of all lines tangent to the graph of $y = \dfrac{6}{x^2+3}$, find the tangent lines of mimimum slope and maximum slope.

8. What is the area of the largest triangle that can be formed in the first quadrant by the x-axis, the y-axis and a tangent line to the graph $y = e^{-x}$?

9. (a) A right circular cylinder is inscribed in a cone with height 6 m and base radius 3 m. Find the largest possible volume of such a cylinder.

   (b) Find the largest volume of a cylinder as in part (a) for a general right circular cone with height h and base radius r.

10. The following equation represents the power P that is required to keep a plane moving at a speed v,

$$P(v) = av^2 + \frac{b}{v^2}$$

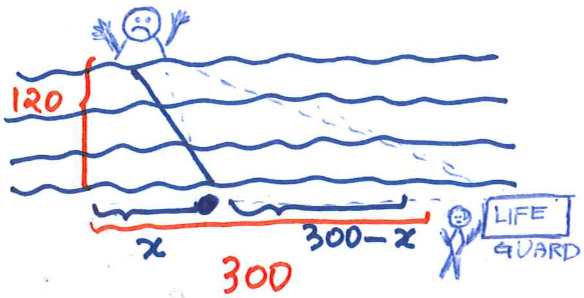
where a and b are positive constants. Determine the speed at which the least power is required to keep the plane moving.

11. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths a cm and b cm if two sides of the rectangle lie along the legs.

12. Car B is 30 miles directly east of Car A and begins moving west at v_B mph. At the same moment car A begins moving north at v_A mph. At what time are the two cars closest?

Q1. (1) Objective function: Time to get to the swimmer.
work sheet

(2)



We want to find the x to enter the water such that it's the quickest way to help.

(3)

Objective function = Total time

= time on sand running + time in the water swimming.

(4) Constraint : Speed on the sand : 13 ft/sec

" in the water : 5 ft/sec

(*) $\text{Speed} = \dfrac{\text{distance}}{\text{time}}$

$\Downarrow$

$\text{time} = \dfrac{\text{distance}}{\text{Speed}}$

$\Downarrow$

$$\begin{cases} \text{Time on sand} = \dfrac{300 - x}{13} \\ \text{time in water} = \dfrac{\text{Hypotenuse}}{5} = \dfrac{\sqrt{120^2 + x^2}}{5} \end{cases}$$

Total time :

$$T = T_{sand} + T_{water}$$

$$T(x) = \frac{300-x}{13} + \frac{\sqrt{120^2+x^2}}{5} \rightarrow (120^2+x^2)^{1/2}$$

Domain : $0 \le x \le 300$

$$T'(x) = \frac{1}{13}(-1) + \frac{1}{5}\left(\cancel{2}x \cdot \frac{1}{\cancel{2}}(120^2+x^2)^{-\frac{1}{2}}\right)$$

$$T'(x) = -\frac{1}{13} + \frac{x}{5\sqrt{(120^2+x^2)}}$$

$\downarrow$ T' undefined

$\Downarrow$

denominator always $+$

$\hookrightarrow$ everywhere defined.

$\downarrow$ $T'(x) = 0$

$\Downarrow$

Common denominator

$$T'(x) = \frac{-5\sqrt{120^2+x^2} + 13x}{65\sqrt{120^2+x^2}} = 0$$

$\Downarrow$

$$-5\sqrt{120^2+x^2} + 13x = 0$$

$$13x = 5\sqrt{120^2+x^2}$$

Solve for x :

$$(13)^2 x^2 = 25\left(120^2 + x^2\right)$$

It's easier to keep $(120)^2$, in case it becomes simplified in the next steps.

$$169\, x^2 = 25 \cdot (120)^2 + 25\, x^2$$

$$169\, x^2 - 25\, x^2 = 25 \cdot (120)^2$$

$$\underbrace{144}_{(12)^2}\, x^2 = 25 \cdot (120)^2$$

$$x^2 = \frac{25 \cdot (120)^2}{(12)^2}$$

$$x = \pm \sqrt{\frac{25 \cdot (120)^2}{(12)^2}} = \underbrace{\pm}_{\text{NOT in the domain}} \frac{5 \cdot \cancel{120}^{\,10}}{\cancel{12}}$$

$$\Rightarrow \boxed{x = 50}$$

Final checks:

(1) Is $x = 50$ a local min ?

Sign chart for T':

easier to compute

x	0	test	50	test	300
T'		$-$	0	$+$	
T		↘	local min	↗	

You can also find $T''(x)$ and find that $T''(50) > 0$ ∪ so it's a local min.

long computation ← 2nd derivative test

(2) Is $x=50$ a global min? Check and evaluate T.

$$0 \le x \le 300$$

$$T(0) = \frac{300}{13} + \frac{\sqrt{120^2}}{5} \approx 47.1 \text{ sec}$$

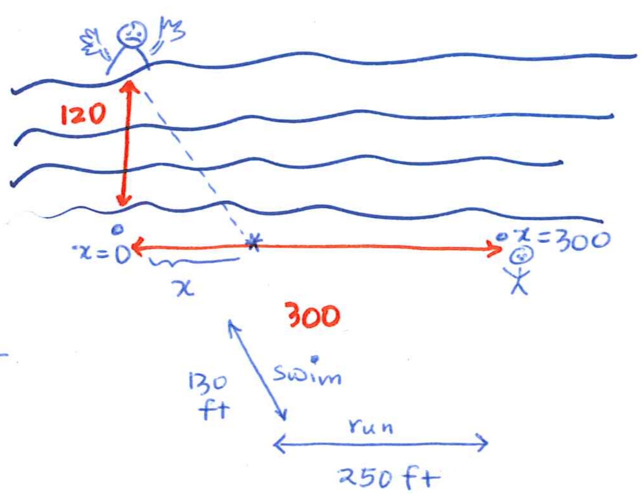
$$T(300) = 0 + \frac{\sqrt{120^2+300^2}}{5} \approx 64.6 \text{ sec}$$

$$T(50) = \frac{250}{13} + \frac{\sqrt{120^2+50^2}}{5} \approx \underbrace{45.2}_{\text{smallest}} \text{ sec}$$

at $x=50$ the function T has a global min.

The person should run for : $300-50=250$ feet.

and should swim for: $\sqrt{120^2+50^2} = 130$ ft



Quickest time

when $x=0 \Rightarrow$ run: 300 ft
Swim: 120 ft
$\Rightarrow$ time $= 47.1$

when $x=300 \Rightarrow$ run : 0
swim: $\sqrt{120^2+300^2}$
longest time $\leftarrow$ $\Rightarrow$ time : 64.6

when $x=50 \Rightarrow$ run : 250 ft
Swim : $\sqrt{120^2+50^2} = 130$ ft
Shortest time $\leftarrow$ $\Rightarrow$ time $= 45.2$