March 28

Lecture 33

Last class : Linear Approx.

Estimate $\sqrt{8.4}$

→ Associate a function to this value :

$$f(x) = \sqrt{x}$$

→ Find a "nice" point close to 8.4 such that $f(x) = \sqrt{x}$ is easily computable.

$$a = 9$$

→ Find the tangent line to $f(x) = \sqrt{x}$ at $a = 9$.

$$\boxed{y = 3 + \frac{1}{6}(x-9)} = L(x)$$

this is the local linearization for $f(x) = \sqrt{x}$ at 9.

→ Evaluate the tangent line, $L(x)$, at the "bad" point.

$$\sqrt{8.4} = f(8.4) \approx L(8.4) = 3 + \frac{1}{6}(8.4-9)$$

$$= 2.9$$

$$\sqrt{8.4} \approx 2.9$$

Example 1. Estimate $e^{0.3}$. "bad" point $x = 0.3$

function: $f(x) = e^x$

A "good nice" point close to 0.3 for which we can evaluate e^x: $a = 0$

Why? b/c $e^0 = 1$

Find the tangent line at $a = 0$ for $f(x) = e^x$.

Slope: $f'(x) = e^x$

$f'(0) = e^0 = 1 \rightsquigarrow m_{tan}$

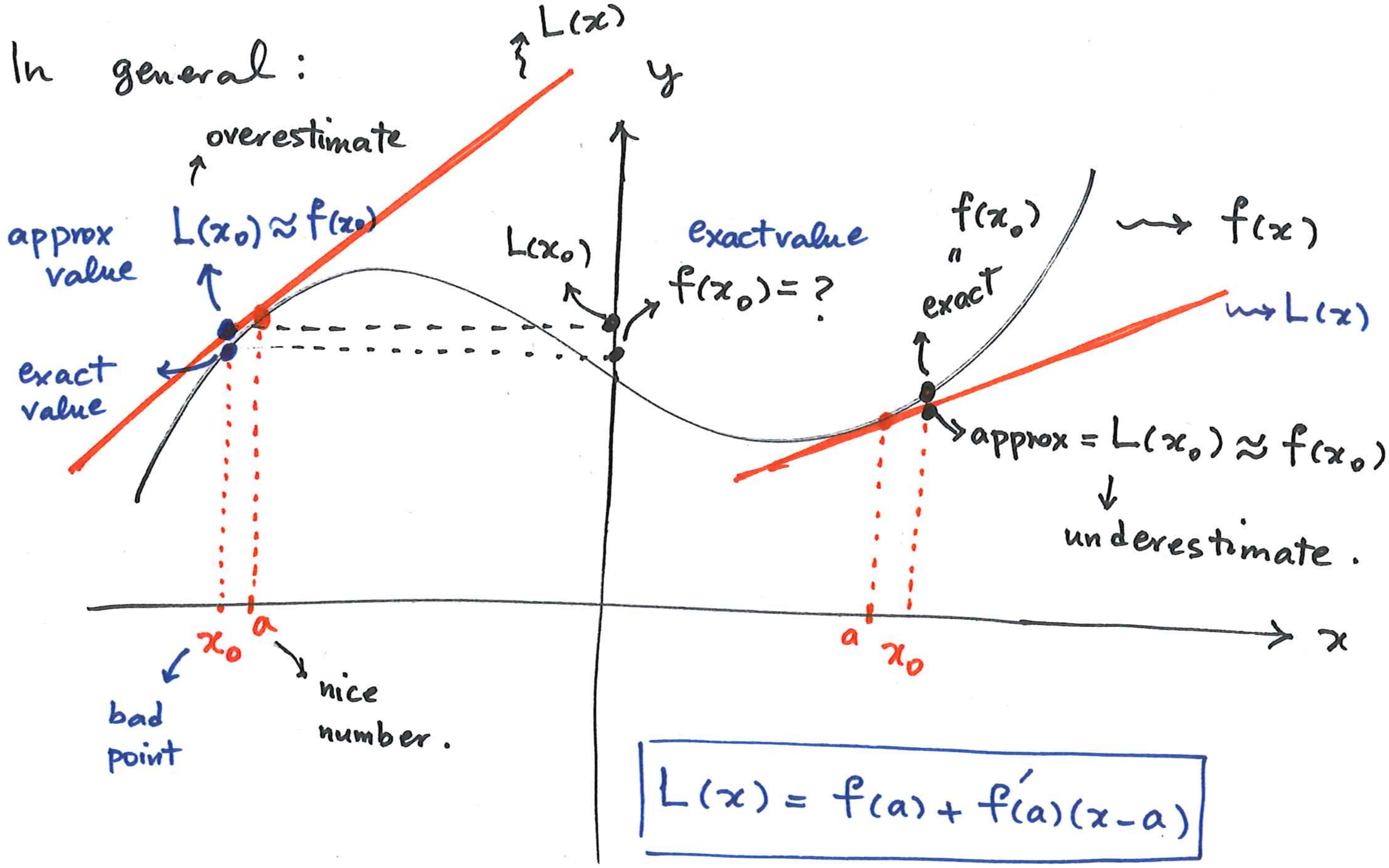
point: $a = 0$, $f(0) = e^0 = 1 \Rightarrow (0, 1)$

$$y - f(a) = f'(a)(x - a)$$

$$y - 1 = 1(x - 0) \Rightarrow \boxed{y = x + 1} = L(x)$$

$$e^{0.3} \approx L(0.3) = 0.3 + 1 = 1.3$$

In general :



$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

* If the approx. value is larger than the exact value , it is an <u>over estimate</u> .

* Similarly if " " " " smaller than " " " " , it is an <u>under estimate</u> .

How to improve our approximation ?

Linear approx $\rightsquigarrow$ Quadratic approx.

add one more term to $L(x)$

Linear Approx for $f(x)$ at "a" :

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

tangent line to $f(x)$ at $x=a$

Quadratic Approx for $f(x)$ at "a" :

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \underbrace{\frac{f''(a)}{2}(x-a)^2}_{\text{extra term added}}}$$

Ex 2. Improve the approx in Ex 1 for $e^{0.3}$.

$$L(x) = x+1$$

$f(x) = e^x \qquad a = 0 \quad , \ f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$$Q(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$Q(x) = 1 + 1 \cdot x + \frac{1}{2} \cdot x^2 = \boxed{1 + x + \frac{x^2}{2}}$$

Now $e^{0.3} \approx Q(0.3) = 1 + 0.3 + \frac{(0.3)^2}{2}$

$$= 1.3 + \frac{0.09}{2}$$

$$= 1.3 + 0.045$$

$$= \underline{1.345}$$

Compare with $L(0.3) = \underline{1.3}$

Exact value $\Rightarrow$ $e^{0.3} = \underline{1.3498}58 \ldots$

Ex 3. Estimate: $\ln(1.1)$ with quadratic approx.

$f(x) = \ln x$

bad point: $x = 1.1$

nice point: $a = 1$ b/c $\ln(1) = 0$

tangent line: $f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1 \rightsquigarrow m_{tan}$

$f(1) = \ln(1) = 0 \Rightarrow (1, 0)$

$$L(x) = f(a) + f'(a)(x-a)$$

$$\overset{a=1}{=} f(1) + f'(1)(x-1)$$

$$= 0 + 1(x-1) \Rightarrow \boxed{L(x) = x-1}$$

With linear approx:

$$\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1$$

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$f'(x) = \frac{1}{x} = x^{-1} \Rightarrow f''(x) = -\frac{1}{x^2}$$

$$\xrightarrow{a=1} f''(1) = -\frac{1}{1} = -1$$

$$Q(x) = x - 1 + \frac{(-1)}{2}(x-1)^2$$

$$\boxed{Q(x) = x - 1 - \frac{(x-1)^2}{2}}$$

$$\ln(1.1) \approx Q(1.1) = \overbrace{1.1 - 1}^{L(1.1)} - \frac{(1.1-1)^2}{2}$$

$$= 0.1 - \frac{(0.1)^2}{2}$$

$$= 0.1 - \frac{0.01}{2}$$

$$= 0.1 - 0.005$$

$$= 0.095$$

Compare with

$\ln(1.1) \approx L(1.1) = \underline{0.1}$, $\underline{Q(1.1) = 0.095}$

Exact value : $\ln(1.1) = \underline{0.0953101} \ldots$