A warm-up exercise before quiz.

Lecture 25
March 9

Example. Sketch the graph of the function

$$f(x) = \frac{1}{x} \cdot \ln\left(\frac{1}{x}\right)$$

a) Find domain, HA and possible VA.

b) find f' and f'', sign charts and sketch the graph.

(a)

Domain. The function has two pieces:

$\frac{1}{x}$ is defined when $x \neq 0$

$\ln\left(\frac{1}{x}\right)$ is defined when $\frac{1}{x} > 0$ so $x > 0$

Combine these two $\rightarrow$ Domain: $x > 0$ or $(0, \infty)$

This means that the graph just goes for the ⊕ x-values.

Possible **VA**: Domain $= (0, \infty)$ suggests that $x = 0$ is the possible VA. We need to verify the limit definition:

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) \ln\left(\frac{1}{x}\right) = \lim_{x \to 0^+} \ln\left(\frac{1}{x}^{\frac{1}{x}}\right) = \infty^{\infty}$$

NOT learned this kind of limits

$x \to 0^-$ NOT possible because the domain is positive x.

But let's assume $\lim_{x \to 0^+} \left(\frac{1}{x}\right) \ln\left(\frac{1}{x}\right) = \infty$ (This is given, you don't need to compute this.)

So $x = 0$ is VA.

HA : To find HA of any function we should find the following limit:

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x).$$ If we get a fixed finite number for any of these limits say "b", then $y=b$ is HA of f.

$$\lim_{x\to\infty} \frac{1}{x}\ln\left(\frac{1}{x}\right) = \frac{1}{\infty}\cdot\overbrace{\ln\left(\frac{1}{\infty}\right)}^{\ln(0)} = 0\times\infty \rightsquigarrow \text{indeterminate}$$

$\rightsquigarrow$ Rewrite f as a fraction

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{1}{x}\right)}{x} = \frac{\ln\left(\frac{1}{\infty}\right)}{\infty} = \frac{\ln(0)}{\infty} = \frac{\infty}{\infty}\ ! \rightsquigarrow \text{indeterminate}$$

$\rightsquigarrow$ Good for l'Hôpital

$$= \lim_{x\to\infty} \frac{\left(\ln\left(\frac{1}{x}\right)\right)'}{(x)'} \xrightarrow{\text{Chain rule}} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cdot\frac{1}{\frac{1}{x}}}{1} = \lim_{x\to\infty} -\frac{1}{x^2}\cdot x$$

$$= \lim_{x\to\infty} -\frac{x}{x^2}$$

$$= \lim_{x\to\infty} -\frac{1}{x} = 0$$

So we found $\lim_{x\to\infty} f(x) = 0$

$\Rightarrow y=0$ is HA of f.

(*) Note $x\to-\infty$ is not possible. The domain is positive numbers, No way to have $x\to-\infty$.

(b) $f(x) = \frac{1}{x}\ln\left(\frac{1}{x}\right)$

$$\Rightarrow f'(x) = -\frac{1}{x^2}\ln\left(\frac{1}{x}\right) + \frac{1}{x}\cdot\overbrace{\left(-\frac{1}{x^2}\cdot\frac{1}{\frac{1}{x}}\right)}^{-\frac{1}{x}}$$

$$= -\frac{1}{x^2}\ln\left(\frac{1}{x}\right) - \frac{1}{x^2}$$

$$\xrightarrow[\text{factor}]{f'(x)=0} -\frac{1}{x^2}\left(\ln\left(\frac{1}{x}\right)+1\right) = 0$$

$\longrightarrow -\frac{1}{x^2} = 0 \xrightarrow{1=0}$ NEVER

$\longrightarrow \ln\left(\frac{1}{x}\right)+1 = 0 \Longrightarrow \ln\left(\frac{1}{x}\right) = -1$

$$\ln\left(\tfrac{1}{x}\right) = -1 \xrightarrow{\;e\;} e^{\ln\left(\frac{1}{x}\right)} = e^{-1} \Rightarrow \frac{1}{x} = \frac{1}{e} \Rightarrow \boxed{x = e}$$

$f'(x)$ is defined for all $x > 0$, so the only critical number is $x = e$

Sign chart :

$$f'(x) = \left(-\frac{1}{x^2}\right)\left(\ln\left(\tfrac{1}{x}\right) + 1\right)$$

always negative

test $\frac{1}{e^2} = e^{-2}$ test e^2 $\rightsquigarrow$ choose points that are easy to evaluate :

x		e	
f'	$-$	0	$+$
f	$\searrow$	local min	$\nearrow$

$$\begin{aligned} f'(e) &= \ominus\left(\ln\left(\tfrac{1}{e^2}\right) + 1\right) \\ &= \ominus\left(\ln e^{-2} - 1\right) \\ &= \ominus\left(-2\ln e - 1\right) \\ &= \ominus\left(-2 + 1\right) > 0 \end{aligned}$$

$$\begin{aligned} f'(e^{-2}) &= \ominus\left(\ln e^{+2} + 1\right) \\ &= \ominus\left(2 + 1\right) < 0 \end{aligned}$$

We used the property : $\ln x^n = n \ln x$ and $\ln e = 1$

y-coordinate of the local min : $f(e) = \frac{1}{e}\ln\left(\frac{1}{e}\right) = \frac{1}{e}\ln e^{-1} = -\frac{1}{e} \Rightarrow \left(e, -\frac{1}{e}\right)$

$\approx (2.71, -0.37)$

$$f'(x) = -\frac{1}{x^2}\left(\ln\left(\tfrac{1}{x}\right) + 1\right)$$

$$\Rightarrow f''(x) = \frac{2}{x^3}\left(\ln\left(\tfrac{1}{x}\right) + 1\right) - \frac{1}{x^2}\Big(\overbrace{-\frac{1}{x^2}\cdot\frac{1}{\frac{1}{x}}}^{-\frac{1}{x}}\Big)$$

$$= \frac{2}{x^3}\left(\ln\left(\tfrac{1}{x}\right) + 1\right) + \frac{1}{x^3} \underset{\text{factor}}{=} \frac{1}{x^3}\left(2\ln\left(\tfrac{1}{x}\right) + 3\right)$$

$\Rightarrow f''(x) = 0 \longrightarrow \frac{1}{x^3} = 0 \longrightarrow$ NEVER

$\longrightarrow 2\ln\left(\frac{1}{x}\right) + 3 = 0 \Rightarrow \ln\left(\frac{1}{x}\right) = -\frac{3}{2} \Rightarrow \frac{1}{x} = e^{-\frac{3}{2}}$

$\Rightarrow x = e^{\frac{3}{2}}$

Go to sign chart for f'' :

x	test $\frac{1}{e}$ ↑	$e^{3/2}$	test e^2 ↗
f''	$+$		$-$
f	$\cup$	INF Point	$\cap$

$$f''(x) = \frac{1}{x^3}\left(2\ln\left(\frac{1}{x}\right)+1\right)$$

since $x>0$

$\frac{1}{x^3}$ is always $\oplus$

$$f''(e^2) = \oplus\left(2\ln\left(\frac{1}{e^2}\right)+1\right)$$
$$= \oplus\left(2\ln e^{-2}+1\right) = \oplus(-4+1) < 0$$

$$f''\left(\frac{1}{e}\right) = \oplus\left(2\ln\left(\frac{1}{\frac{1}{e}}\right)+1\right) = \oplus(2\ln e+1)$$
$$= \oplus(2+1) > 0$$

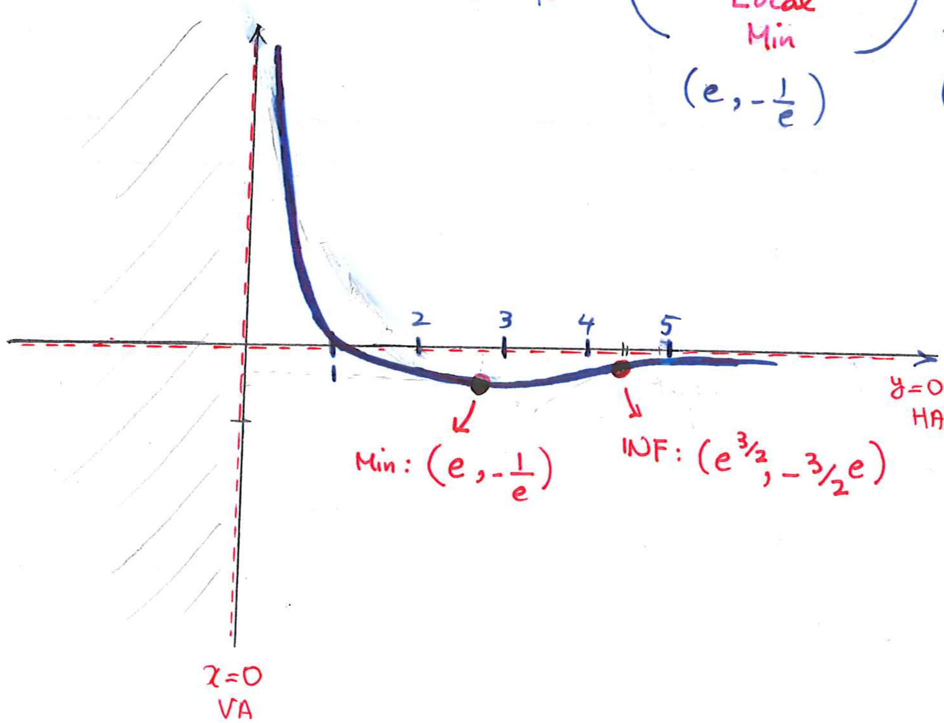
y-coordinate : $x = e^{\frac{3}{2}}$

$$f\left(e^{\frac{3}{2}}\right) = \frac{1}{e^{\frac{3}{2}}}\ln\left(\frac{1}{e^{\frac{3}{2}}}\right) = e^{-\frac{3}{2}}\ln\left(e^{-\frac{3}{2}}\right) = e^{-\frac{3}{2}}\cdot\frac{-3}{2}\ln e$$
$$= -\frac{3}{2}e^{-3/2}$$

$$\Rightarrow \left(e^{3/2}, -\tfrac{3}{2}e^{-3/2}\right) \approx (4.48, -0.33)$$

Combined Summary Chart :

x	0		e		$e^{3/2}$		∞
f'		$-$ ↘		$+$ ↗	0	$+$ ↗	
f''		$+$ $\cup$	0	$+$ $\cup$		$-$ $\cap$	
f			Local Min $\left(e, -\frac{1}{e}\right)$		INF Point $\left(e^{3/2}, -\frac{3}{2}e^{-3/2}\right)$		



NOTE :

x-int : $\frac{1}{x}\ln\left(\frac{1}{x}\right) = 0$

$\Rightarrow \ln\left(\frac{1}{x}\right) = 0$

$\Rightarrow \frac{1}{x} = e^0 = 1$

$\Rightarrow x = 1$