Curve Sketching (cont'd)

Lecture 24
Mar 7

Today, we continue with graphing more complicated functions.
First, let's sketch the graph of another rational function.

Example 1. Sketch the function $y = \dfrac{x^2}{1-x^2}$.

1st step: f itself

Domain: Exclude where $1-x^2=0 \Rightarrow x^2=1$
$\Rightarrow x=1, \; x=-1$

$\Rightarrow$ Domain: Everywhere except $x=1$ and $x=-1$

This suggests that $x=1,-1$ are the possible VA. Now we verify:

$$\lim_{x\to1^+}\frac{x^2}{1-x^2} = \lim_{x\to1^+}\frac{x^2}{(1-x)(1+x)} = \frac{1}{\underbrace{(1-1^+)}_{0^-}(1+1)} = -\infty$$

$\Rightarrow x=1$ VA

$$\lim_{x\to1^-}\frac{x^2}{1-x^2} = \lim_{x\to1^-}\frac{x^2}{(1-x)(1+x)} = \frac{1}{\underbrace{(1-1^-)}_{0^+}(1+1)} = +\infty$$

$$\lim_{\substack{x\to-1^+ \\ \swarrow \\ -0.9}}\frac{x^2}{(1-x)(1+x)} = \frac{(-1)^2}{(1-(-1))\underbrace{(1+(-1)^+)}_{1-0.9=0^+}} = \frac{1}{0^+} = +\infty$$

$\Rightarrow x=-1$ VA

$$\lim_{\substack{x\to-1^- \\ \swarrow \\ -1.1}}\frac{x^2}{(1-x)(1+x)} = \frac{(-1)^2}{(1-(-1))\underbrace{(1+(-1)^-)}_{1-1.1=0^-}} = \frac{1}{0^-} = -\infty$$

HA is when $\lim_{x\to\pm\infty} f(x)$ is a fixed number. Let's verify that

$$\lim_{x\to\infty}\frac{x^2}{1-x^2} = \frac{\infty}{-\infty} \xrightarrow[\text{Rule}]{\text{l'Hôpital}} = \lim_{x\to\infty}\frac{\cancel{2x}}{-\cancel{2x}} = -1$$

↗ Curve approaches -1 from negative side because of $\frac{\infty}{-\infty}$

Similarly : $\lim\limits_{x \to -\infty} \dfrac{x^2}{1-x^2} = -1 \Rightarrow y=-1$ is HA .

$\hookrightarrow$ here also the curve approach $y=-1$ from negative side $\dfrac{(-\infty)^2}{-(-\infty)^2}<$

Now let's check the intercepts :

$f(x) = \dfrac{x^2}{1-x^2}$ $\rightsquigarrow$ y-int $\xrightarrow{x=0}$ $f(0) = \dfrac{0^2}{1-0^2} = 0 \Rightarrow (0,0)$

$\rightsquigarrow$ x-int $\xrightarrow{y=0}$ numerator $= 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

2^{nd} step : f' (INC/DEC)

$$f'(x) = \frac{2x(1-x^2)-(-2x)(x^2)}{(1-x^2)^2} = \frac{2x - \cancel{2x^3} + \cancel{2x^3}}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

So $f'(x)$ is also undefined when $x=1$ and $x=-1$

and $f'(x) = 0$ when $2x = 0 \Rightarrow x = 0$

Sign Chart for f'

	test -2		test $-\frac{1}{2}$		test $\frac{1}{2}$		test 2
x	$-\infty$	-1		0		1	$+\infty$
f'	$-$	∞	$-$	0	$+$	∞	$+$
f	$\searrow$	VA	$\searrow$	local min	$\nearrow$	VA	$\nearrow$

$\hookrightarrow$ y-coordinate :

$f(0) = 0 \Rightarrow (0,0)$ is the local min point.

test points :

$f'(x) = \dfrac{2x}{(1-x^2)^2}$ always $\oplus$

$\rightsquigarrow f'(2) = \dfrac{4}{\oplus} > 0$

$f'(\frac{1}{2}) = \dfrac{2 \times \frac{1}{2}}{\oplus} > 0$

$f'(-\frac{1}{2}) = \dfrac{2 \times (-\frac{1}{2})}{\oplus} < 0$

$f'(-2) = \dfrac{2 \times (-2)}{\oplus} < 0$

3^{rd} step : f'' (concavity)

$$f''(x) = \frac{2(1-x^2)^2 - \underline{2x(2(1-x^2))(-2x)}}{(1-x^2)^4} \overset{\text{factor}}{=}$$

$$f''(x) = \frac{\cancel{(1-x^2)}\left(2(1-x^2)+8x^2\right)}{(1-x^2)^{\cancel{4}3}} = \frac{2+6x^2}{(1-x^2)^3}$$

f'' is also undefined at $x = 1, -1$ and $f''(x) = 0$ when

$$2 + 6x^2 = 0 \Rightarrow 6x^2 = -2 \Rightarrow x^2 = -\frac{2}{6} \quad \text{NOT possible}$$

Sign chart for f'' →

x	$-\infty$ (test (-2))	-1	(test 0)	1	(test 2) ∞
f''	$-$		$+$		$-$
f	$\cap$	VA	$\cup$	VA	$\cap$

Although the concavity changes around 1 and (-1), they are NOT inflection points

$$f''(x) = \frac{2+6x^2}{(1-x^2)^3}$$

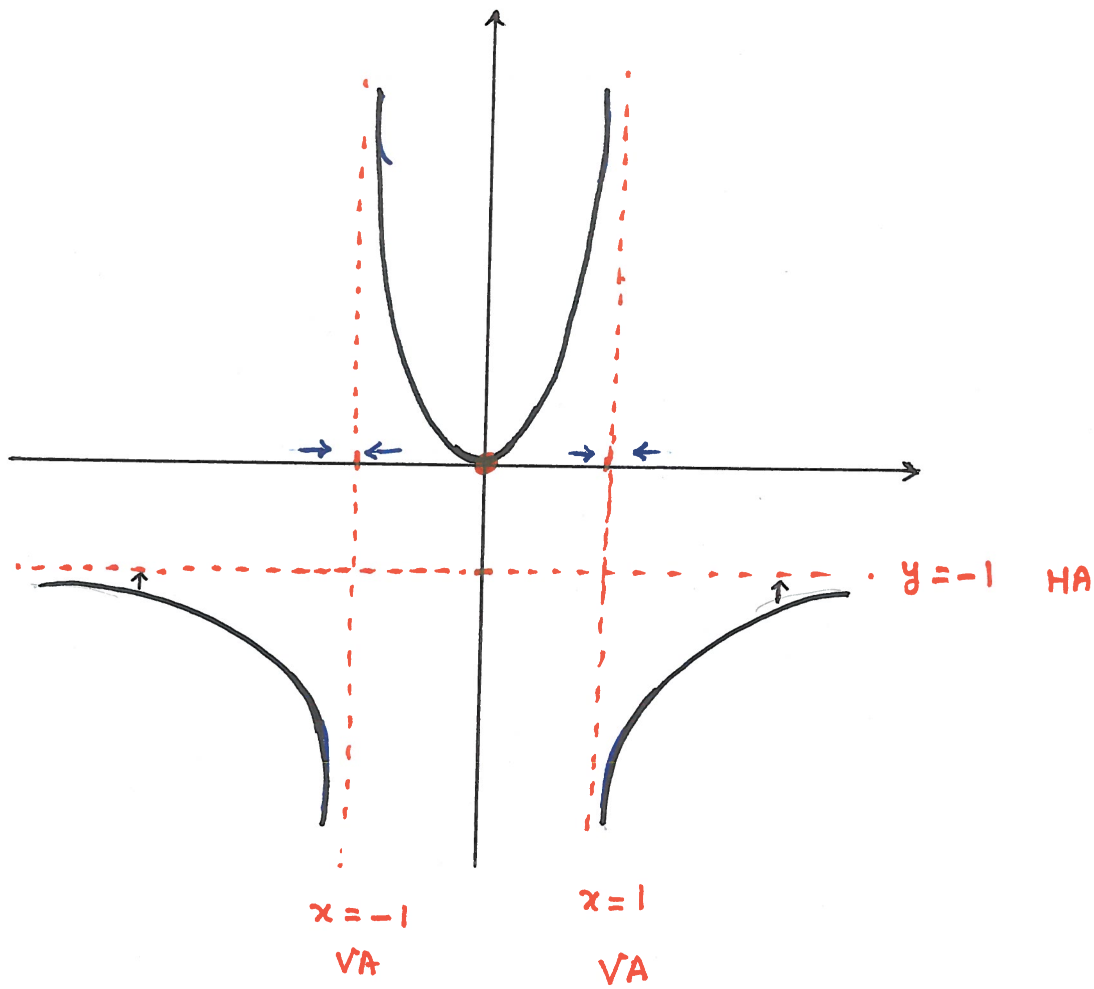
always $\oplus$

$$\rightsquigarrow f''(2) = \frac{\oplus}{(1-4)^3} = \frac{\oplus}{\ominus} < 0$$

$$f''(0) = \frac{\oplus}{(1-0)^3} > 0 \quad , \quad f''(-2) = \frac{\oplus}{(1-4)^3} < 0$$

4th step. Summary Chart

x	$-\infty$	-1		0		1	∞
f'	$-$ ↘		$-$ ↘		$+$ ↗		$+$ ↗
f''	$-$ $\cap$		$+$ $\cup$		$+$ $\cup$		$-$ $\cap$
f		VA		local Min		VA	



y = -1 HA
x = -1
VA
x = 1
VA

Example 2. Sketch the graph of the function

$$f(x) = x e^{-2x^2}$$

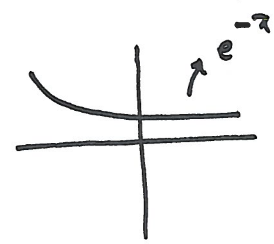
$\boxed{f \text{ itself}}$:

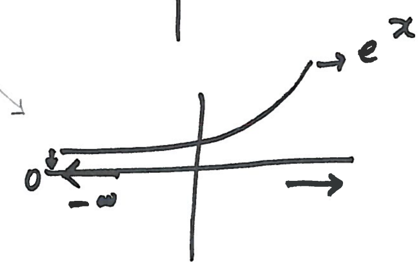
Domain: $\mathbb{R} \rightsquigarrow$ NO VA

Intercepts:
$$\begin{cases} y\text{-int} \xrightarrow{x=0} f(0) = 0e^{0} = 0 \\ x\text{-int} \xrightarrow{y=0} x\underbrace{e^{-2x^2}}_{\text{always } \oplus} = 0 \Rightarrow x = 0 \end{cases} \Rightarrow (0,0)$$

HA:
$$\lim_{x\to\infty} x e^{-2x^2} = \infty e^{-2\infty} = \infty \underbrace{e^{-\infty}}_{0} = \infty \times 0$$

$\downarrow$ indeterminate limit.





Rewrite negative power

$$= \lim_{x\to\infty} \frac{x}{e^{2x^2}} = \frac{\infty}{e^{\infty}} = \frac{\infty}{\infty}$$

$\downarrow$ l'Hôpital

l'Hôpital
$$= \lim_{x\to\infty} \frac{1}{4x e^{2x^2}} = \frac{1}{\infty} = 0$$

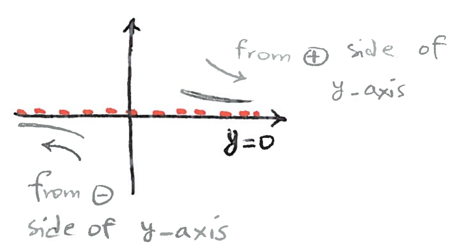
In fact:

$$\lim_{x\to+\infty} \frac{1}{4xe^{2x^2}} = \frac{1}{+\infty} = 0 \rightsquigarrow \text{Approaches } 0 \text{ from } \oplus \text{ side}$$

$$\Rightarrow \boxed{y = 0 \quad \text{HA}}$$

and

$$\lim_{x\to-\infty} \frac{1}{4xe^{2x^2}} = \frac{1}{-\infty} = 0 \rightsquigarrow \text{Approaches } 0 \text{ from } \ominus \text{ side}$$



$\boxed{f'}$: INC/DEC $\qquad f(x) = x e^{-2x^2}$

$$f'(x) = 1 \cdot e^{-2x^2} + x(-4x) e^{-2x^2}$$

↑ Product rule

$$= e^{-2x^2}(1 - 4x^2)$$

$f'(x) = 0$ $\qquad\qquad$ $f'(x)$ $\underbrace{\text{undefined}}$

NEVER

$\Downarrow$

$e^{-2x^2} = 0 \longrightarrow$ NEVER , always $\oplus$

, $1 - 4x^2 = 0 \Rightarrow 4x^2 = 1$

$\Rightarrow x^2 = \frac{1}{4} \Rightarrow \boxed{x = \frac{1}{2}, \; x = -\frac{1}{2}}$

x	$-\infty$	↑ -1	$-\frac{1}{2}$	↑ 0	$\frac{1}{2}$	↑ 1	∞
f'		$-$	0	$+$	0	$-$	
f		↘	local min	↗	local max	↘	

$f'(x) = e^{-2x^2}(1 - 4x^2)$

e^{-2x^2} → always $\oplus$

y-coordinate: $\quad f(x) = x e^{-2x^2}$

$f(\frac{1}{2}) = \frac{1}{2} e^{-2 \times \frac{1}{4}} = \frac{1}{2} e^{-\frac{1}{2}} \approx 0.3$

$e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

$\boxed{\begin{array}{l} (\frac{1}{2}, 0.3) \rightarrow \text{Max} \\ (-\frac{1}{2}, -0.3) \rightarrow \text{Min} \end{array}}$

$e \approx 2.71 \Rightarrow \sqrt{e} \approx 1.4 \Rightarrow \frac{1}{1.4} \approx 0.7 \qquad \frac{1}{2} \times 0.7$

$\boxed{f''}$: Concavity

$$f'(x) = e^{-2x^2}(1-4x^2)$$

$$f''(x) = e^{-2x^2}(-12x + 16x^3)$$

$f''(x) = 0$

defined everywhere

$\Downarrow$

$e^{-2x^2} = 0$ NEVER

$$-12x + 16x^3 = 0$$

$$4x(-3 + 4x^2) = 0$$

$$\Rightarrow \quad x = 0$$

$$-3 + 4x^2 = 0 \Rightarrow 4x^2 = 3$$

$$\Rightarrow x^2 = \frac{3}{4} \Rightarrow x = \sqrt{\frac{3}{4}}, \quad x = -\sqrt{\frac{3}{4}}$$

$$\Rightarrow x = \frac{\sqrt{3}}{2}, \quad x = -\frac{\sqrt{3}}{2}$$

≈ 0.85 $\quad x \approx -0.85$

Good to Remember these:

$\sqrt{3} \approx 1.7$

$\sqrt{2} \approx 1.4$

$\sqrt{5} \approx 2.2$

(*) In the exam, you'll have easy numbers to calculate or they provide you with the approximate values.

x	$-\infty$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	∞
f''		$-$	0	$+$	0	$-$	0	$+$	
f		$\cap$	INF	$\cup$	INF	$\cap$	INF	$\cup$	

$$f''(x) = e^{-2x^2}(-12x + 16x^3)$$

$e^{-2x^2} \to \oplus$

y-coordinates:

≈ 0.85

$f\left(-\frac{\sqrt{3}}{2}\right) \approx -0.2 \Rightarrow \left(-\sqrt{3}/2, -0.2\right)$

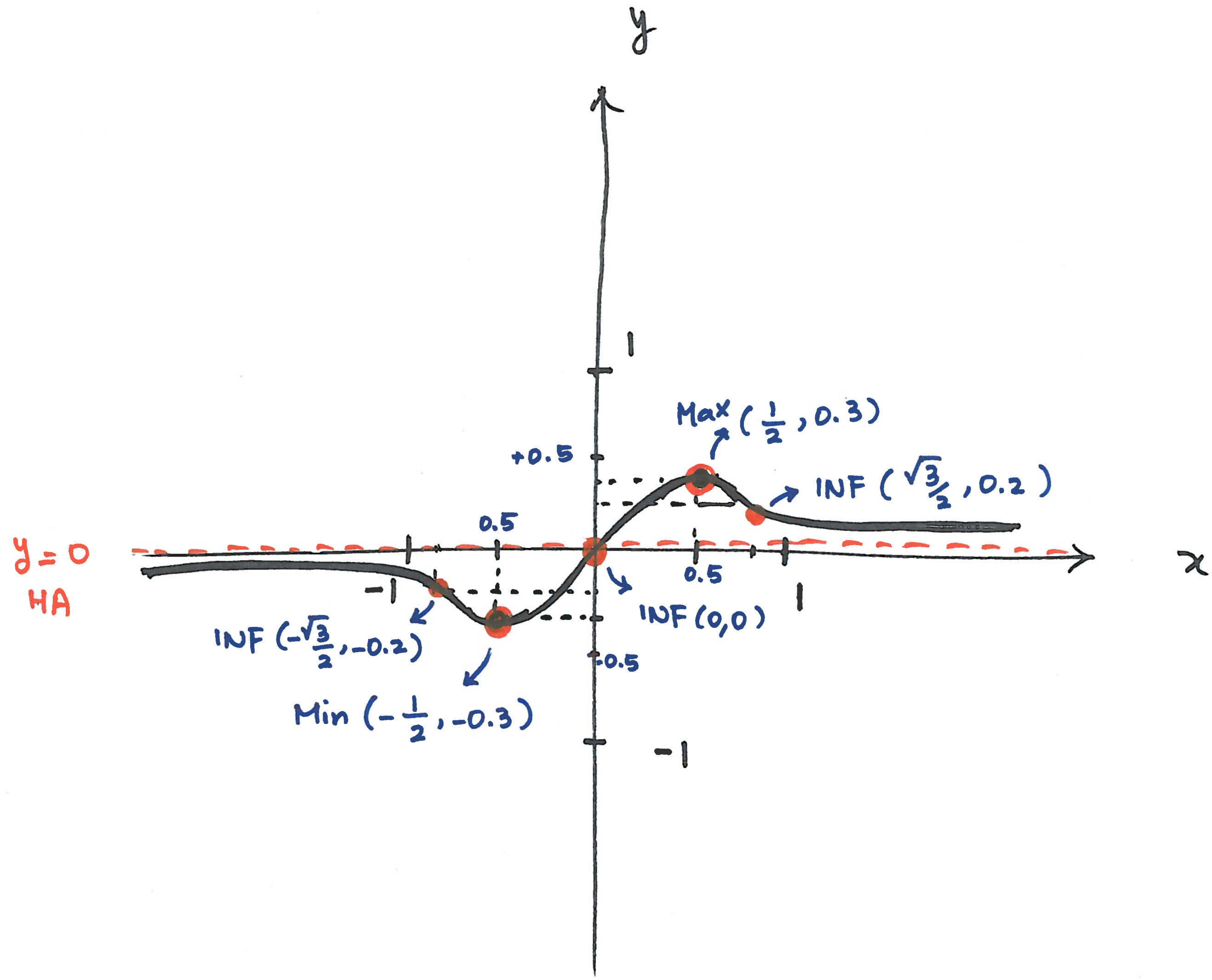
$f\left(\frac{\sqrt{3}}{2}\right) \approx 0.2 \qquad \left(\sqrt{3}/2, 0.2\right)$

$f(0) = 0 \qquad (0, 0)$

INF points

Summary Chart:

x	$-\infty$		$-\frac{\sqrt{3}}{2}$		$-\frac{1}{2}$		0		$\frac{1}{2}$		$\frac{\sqrt{3}}{2}$		∞
f'		$-$ $\searrow$		$-$ $\searrow$	0	$+$ $\nearrow$		$+$ $\nearrow$	0	$-$ $\searrow$		$-$ $\searrow$	
f''		$-$ $\cap$	0	$+$ $\cup$		$+$ $\cup$	0	$-$ $\cap$		$-$ $\cap$	0	$+$ $\cup$	
f			INF		MIN		INF		MAX		INF		



Exercise. Sketch $y = \frac{3}{2}\sqrt[3]{x^2} + x$.