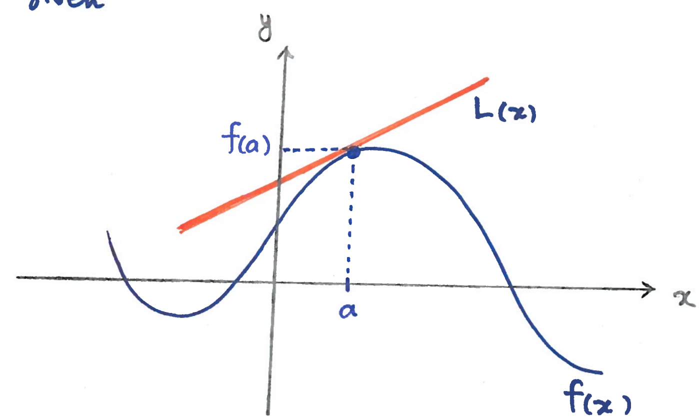
Linear Approximation — Local Linearization

March 26
Lecture 32

Approximate a function (usually complicated) with a line.

⇝ What is this line? The tangent line

Recall ⇝ Tangent line: The function $f(x)$ and a point $(a, f(a))$ on f are given



$(a, f(a))$ is the touch point, both on the graph and on the line. To write the equation of the tangent line we need: slope ⇝ $m_{tan} = f'(a)$

& point ⇝ $(a, f(a))$

$$\Rightarrow \quad y - f(a) = f'(a)(x-a)$$

or

$$\boxed{y = \underbrace{f'(a)(x-a) + f(a)}_{\text{call this } L(x)}}$$ ⟶ tangent line equation

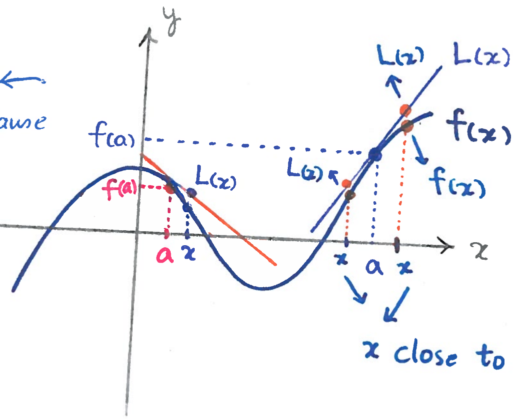
then $L(x) = f'(a)(x-a) + f(a)$ is the tangent line equation.

Linear Approximation :

If f is differentiable at "a" and x is a point close to "a", then the function $f(x)$ can be approximated with its tangent line $L(x)$.

Or $f(x)$ is close to $L(x) \rightsquigarrow f(x) \approx L(x)$

This is an example of over-estimate because the value of $L(x)$ is greater than $f(x)$
$\Downarrow$
$L(x)$ sits above $f(x)$.



$$\boxed{\begin{array}{l} f(x) \approx L(x) \\ L(x) = f'(a)(x-a) + f(a) \end{array}}$$

(*) Instead of evaluating f at x, we evaluate L at x and estimate f with L.

(*) In linearization questions, we should first find :

"a" : The touch point $\rightsquigarrow$ The "good" point usually with no decimal points.

"x" : The point close to "a" $\longrightarrow$ usually has decimals.

f : The function that we should use to find the linearization. Sometimes it's explicitly given in the question, but sometimes we should find it from the given estimate.

Example. a) Find the linear approximation of $f(x)=\sqrt{x}$ at $x=9$

tangent line at $x=9$

Find the touch point $\xrightarrow{x=9}$ $f(9)=\sqrt{9}=3 \rightsquigarrow (9,3)$

slope $\rightsquigarrow f'(9)$

$$f(x)=x^{\frac{1}{2}} \quad \Rightarrow f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}} \Rightarrow f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$$

$$\Rightarrow \quad y-3=\frac{1}{6}(x-9)$$

$$\Rightarrow \quad \boxed{L(x)=\frac{1}{6}(x-9)+3} \longrightarrow \text{linear approximation of } f(x)=\sqrt{x}$$

b) Sketch the graph of f and its local linearization.

$\rightsquigarrow f(x)=\sqrt{x}$

$f(0)=\sqrt{0}=0$

$f(1)=\sqrt{1}=1$

$f(4)=\sqrt{4}=2$

$f(9)=\sqrt{9}=3$

$\rightsquigarrow L(x)=\frac{1}{6}(x-9)+3$

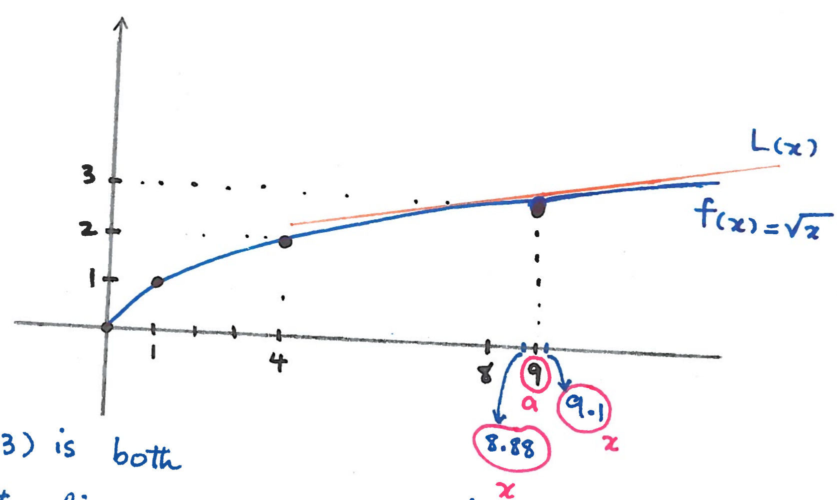
$f(9)=3$

$\Rightarrow$ $(9,3)$ is both on the line and on $f(x)=\sqrt{x}$

$$f(6)=\frac{1}{6}(\overset{-3}{\cancel{6-9}})+3$$

$$=-\overset{-0.5}{\cancel{\frac{1}{2}}}+3$$

$$=2.5$$



The tangent line always on the touch point, we don't change the tangent line, we use the same line to evaluate and estimate for x's close to the touch point "a".

c) find $\sqrt{9.1}$ and $\sqrt{8.88}$.

First find : "a", "x" and $f(x)$.

$f(x)$ is given : $f(x) = \sqrt{x}$

$\sqrt{9.1} = f(9.1)$ ⇝ 9.1 is close to 9

"x" : the point close to "a" usually with decimal points.

"a" : the "nice" point with no decimal points.

$$\Rightarrow \quad f(x) \approx L(x)$$

$$\Rightarrow \sqrt{9.1} = f(9.1) \approx L(9.1)$$

from part (a) : $L(x) = \frac{1}{6}(x-9)+3$

$$\Rightarrow L(9.1) = \frac{1}{6}(9.1-9)+3 = \frac{0.1}{6}+3 = 3.0166$$

$$\Rightarrow \boxed{\sqrt{9.1} \approx 3.0166}$$

Similarly : $\sqrt{8.88} = f(8.88) \approx L(8.88)$

$$= \frac{1}{6}(8.88-9)+3$$

$$= \frac{-0.12}{6}+3$$

$$= -0.02+3 = 2.98$$

$$\Rightarrow \boxed{\sqrt{8.88} \approx 2.98}$$

Example Estimate $\ln(1.2)$ and $\ln(e - 0.05)$

Find f, "a" and "x".

f is NOT given in the question, but since we are looking for "ln" of some number so: $\underline{f(x) = \ln x}$

"a" is the "nice" touch point : $a = 1$

"x" is the point close to "a" : $x = 1.2$

$\Rightarrow$ We use linear approximation to estimate the values, so first we need to find the equation to the line.

$f(x) = \ln x$

$a = 1 \Rightarrow f(1) = \ln 1 = 0 \Rightarrow (1, 0)$ touch point

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} \Rightarrow m_{tan} = 1$

$\Rightarrow y - 0 = 1(x - 1)$

$\Rightarrow \boxed{L_1(x) = x - 1}$

1.2 is close to 1, so we can use $L_1(x)$ to estimate:

$\ln(1.2) = f(1.2) \approx L_1(1.2) = 1.2 - 1 = 0.2$

$\Rightarrow \boxed{\ln(1.2) \approx 0.2}$

$\ln(e - 0.05)$: We can NOT use the previous $L(x)$, because the point "a" is different.

Again : $f(x) = \ln x$, But $a = e$ and $x = e - 0.05$

touch point $a = e$ $\xrightarrow{f(e)}$ $\ln e = 1 \Rightarrow (e, 1)$

slope : $f'(e) = \frac{1}{e} \Rightarrow y - 1 = \frac{1}{e}(x - e)$

$$\Rightarrow \boxed{L_2(x) = \frac{1}{e}(x - e) + 1}$$

$e - 0.05$ is close to e, we use $L_2(x)$ to estimate :

$$\ln(e - 0.05) = f(e - 0.05) \approx L_2(e - 0.05)$$

$$= \frac{1}{e}(\cancel{e} - 0.05 - \cancel{e}) + 1$$

$$= \boxed{\frac{-0.05}{e} + 1}$$

Example. a) Find the linearization to the function

$$g(x) = x\, f(x^2) \quad \text{at} \quad x=2$$

given that $f(2)=-1$, $f'(2)=6$, $f(4)=3$, $f'(4)=-4$

b) estimate $g(1.99)$

a) linearization : tangent line at $x=2$

touch point : $g(2) = 2\, f(2^2) = 2 f(4) = 2\times 3 = 6 \rightarrow (2,6)$

slope : $g'(2)$

$$g'(x) = \underbrace{1\times f(x^2)}_{\text{product Rule}} + x \times \overbrace{\underbrace{f'(x^2)}_{\text{outside}} \times \underbrace{2x}_{\text{inside}}}^{\text{Chain rule}}$$

$$\Rightarrow g'(2) = f(2^2) + 2\, f'(2^2) \times 2 \times 2$$

$$= 3 + 2 \times -4 \times 4 = 3 - 32 = -29$$

$$\Rightarrow y - 6 = -29\,(x-2) \Rightarrow \boxed{L(x) = -29\,(x-2) + 6}$$

I'd rather NOT to distribute -29 because when evaluating $L(1.99)$ computations will be easier.

b) 1.99 is close to 2 → Use $L(x)$ to estimate :

$$g(1.99) \approx L(1.99) = -29\,(1.99-2)+6$$

$$= -29 \times -0.01 + 6$$

$$= +0.29 + 6$$

$$= \boxed{6.29}$$

Practice. Use local linearization to estimate

(a) $(2.02)^8$

(b) $\sqrt[3]{0.98}$

(c) $e^{0.06}$

(a) $(2.02)^8$ what is f, a and x ?

Something is raised to power 8 $\Rightarrow f(x) = x^8$

$a = 2$ the "nice" touch point

$x = 2.02$.

$f(x) = x^8 \Rightarrow f(2) = 2^8 \Rightarrow (2, 2^8)$ touch point

$f'(x) = 8x^7 \Rightarrow f'(2) = 8 \cdot 2^7 = m_{tan}$

$$\Rightarrow \quad y - 2^8 = 8 \times 2^7 (x - 2)$$

$$\Rightarrow \quad \boxed{L(x) = 8 \times 2^7 (x-2) + 2^8}$$

2.02 is close to 2, we use $L(x)$ to estimate:

$$(2.02)^8 = f(2.02) \approx L(2.02)$$
$$= 8 \times 2^7 (2.02 - 2) + 2^8$$
$$= 2^3 \times 2^7 \times 0.02 + 2^8$$
$$= 20.48 + 256 = \underline{276.48}$$

(b) $\sqrt[3]{0.98}$ $\rightsquigarrow$ taking the cubic root $\rightsquigarrow$ $f(x) = \sqrt[3]{x}$

$a = 1$, $x = 0.98$

$f(x) = x^{\frac{1}{3}} \Rightarrow f(1) = \sqrt[3]{1} = 1 \rightarrow (1,1)$ touch point

$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3\sqrt[3]{x^2}}$

$\Rightarrow f'(1) = \frac{1}{3} \Rightarrow y - 1 = \frac{1}{3}(x-1)$

$\Rightarrow \boxed{L(x) = \frac{1}{3}(x-1) + 1}$

0.98 is close to 1 $\Rightarrow \sqrt[3]{0.98} = f(0.98) \approx L(0.98)$

$L(0.98) = \frac{1}{3}(0.98 - 1) + 1 = \underline{\frac{-0.02}{3} + 1 \approx \sqrt[3]{0.98}}$

(c) $e^{0.06}$ $\rightsquigarrow$ $f(x) = e^x$, 0.06 is close to 0 , therefore

$a = 0$ and $x = 0.06$

$f(0) = e^0 = 1 \Rightarrow (0,1)$ touch point

$f'(x) = e^x \Rightarrow m_{tan} = f'(0) = e^0 = 1 \Rightarrow y - 1 = 1(x-0)$

$\Rightarrow \boxed{L(x) = x + 1}$

$e^{0.06} = f(0.06) \approx L(0.06) = 0.06 + 1 = 1.06$

$\Rightarrow \underline{e^{0.06} \approx 1.06}$