# l'Hôpital's Rule

Lecture 20
Feb 26

Recall from term 1, when we learned how to find limits we saw examples where $\frac{0}{0}$ appeared after direct substitution.

For example; $\lim\limits_{x \to 2} \frac{x^2-4}{x^2+x-6} = \frac{2^2-4}{2^2+2-6} = \frac{0}{0}$ !

We learned that in order to deal with these cases we have to manipulate the numerator and the denominator to get rid of $\frac{0}{0}$. The trick was usually to factor top and bottom.

Rewrite the limit and factor:

$$\lim_{x \to 2} \frac{x^2-4}{x^2+x-6} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{(x+3)\cancel{(x-2)}} = \frac{2+2}{2+3} = \frac{4}{5} \checkmark$$

Now let's check the following limits:

→ $\lim\limits_{x \to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0}$ !

→ $\lim\limits_{x \to 0} \frac{e^x - 1}{2x} = \frac{e^0-1}{2 \times 0} = \frac{0}{0}$ !

→ $\lim\limits_{x \to 1} \frac{5 \ln x}{x-1} = \frac{5 \ln 1}{1-1} = \frac{0}{0}$ !

They are all $\frac{0}{0}$ cases BUT none of them can be factored.

What's the trick?

## l'Hôpital's Rule

This rule helps us to find $\frac{0}{0}$ limits in general.

$\boxed{\frac{0}{0} \text{ is an example of indeterminate limit.}}$

Statement of the rule is the following :

**l'Hôpital's Rule :**

Consider a function is the quotient form $\frac{f(x)}{g(x)}$ where

- $f$ and $g$ are differentiable at a given point $x=a$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$
- and $g'(x) \neq 0$ except possibly at $x=a$ ( $g'(a)=0$ is possible)

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Roughly, this rule approximates functions $f$ and $g$ by the tangent lines to $f$ and $g$ at $x=a$, and then the limits of derivatives are evaluated.

l'Hôpital is the name of a french mathematician who lived in the 17-th century. Based on Wikipedia, the rule was first introduced by another mathematician Johann Bernouli, but this rule is mainly attributed to l'Hôpital !

"A Historical Note"

Guillaume de l'Hôpital

| | |
|---|---|
| **Born** | 1661<br>Paris, France |
| **Died** | 2 February 1704<br>Paris, France |
| **Nationality** | French |

Let's revisit the indeterminate limits given at the beginning of the lecture and see how l'Hôpital's rule helps.

**Example 1.** $$\lim_{x\to 2} \frac{\overbrace{x^2-4}^{f(x)}}{\underbrace{x^2+x-6}_{g(x)}} = \frac{0}{0}$$

$$\left.\begin{array}{l} f \text{ and } g \text{ are both differentiable at } x=2 \\ g'(x) = 2x+1 \text{ is NOT identically equal to } 0 \\ \text{and } \displaystyle\lim_{x\to 2}\frac{f(x)}{g(x)} = \frac{0}{0}\ ! \end{array}\right\} \Rightarrow \text{l'Hôpital's Rule applies.}$$

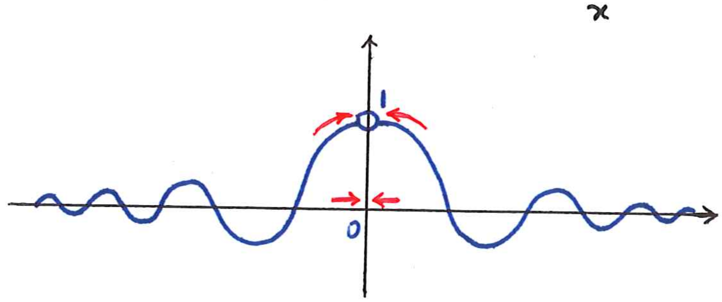
$$\lim_{x\to 2} \frac{(x^2-4)}{(x^2+x-6)} = \lim_{x\to 2} \frac{(x^2-4)'}{(x^2+x-6)'} = \lim_{x\to 2} \frac{2x}{2x+1} = \frac{2\times 2}{2\times 2+1} = \frac{4}{5} \checkmark$$

replacing top and bottom by their derivatives.

**Example 2.** $$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0}\ !$$

l'Hôpital
$$= \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos(0)}{1} = \frac{1}{1} = 1 \checkmark$$

Recall the "cool" graph of $y = \dfrac{\sin x}{x}$




We can also find from the graph that as $x \to 0$, $\dfrac{\sin x}{x}$ approaches 1

$$\boxed{\text{Recall in previous class we found } \lim_{x\to\infty} \frac{\sin x}{x} = 0}$$

**Example 3.** $\lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{e^0 - 1}{2\times 0} = \frac{1-1}{0} = \frac{0}{0}!$

l'Hôpital's Rule $= \lim_{x\to 0} \frac{(e^x-1)'}{(2x)'} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$ ✓

**Example 4.** $\lim_{x\to 1} \frac{5\ln x}{x-1} = \frac{5\ln(1)}{1-1} = \frac{0}{0}!$

l'Hôpital's Rule $= \lim_{x\to 1} \frac{(5\ln x)'}{(x-1)'} = \lim_{x\to 1} \frac{5\cdot\frac{1}{x}}{1} = \frac{5\cdot\frac{1}{1}}{1} = 5$ ✓

**Example 5.** $\lim_{x\to 0} \frac{\cos(3x)-1}{x^2} = \frac{\cos(0)-1}{0^2} = \frac{1-1}{0} = \frac{0}{0}!$

l'Hôpital $= \lim_{x\to 0} \frac{3(-\sin(3x))}{2x} = \frac{-3\sin 0}{2\times 0} = \frac{0}{0}!$

One step of l'Hôpital's rule does NOT make $\frac{0}{0}$ disappear.

⟶ Apply l'Hôpital's Rule once more.

$= \lim_{x\to 0} \frac{-3(3\cos(3x))}{2} = \frac{-9\overset{1}{\cos(0)}}{2} = \frac{-9}{2}$ ✓

As you see, there are indeterminate limits for which the l'Hôpital's rule should be applied multiple times.

**Exercise**: $\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} = ?$ Answer Key $= 6$

Another form of an indeterminate limit is $\frac{\infty}{\infty}$.

l'Hôpital's rule can be applied to this case too :

## l'Hôpital's Rule (Stronger form)

Consider a quotient function $\frac{f(x)}{g(x)}$ where $f$ and $g$ are differentiable in an interval that contains a given number $x=a$ and $g'(x) \neq 0$ except possibly at $x=a$.

If $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ or $-\frac{\infty}{\infty}$

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

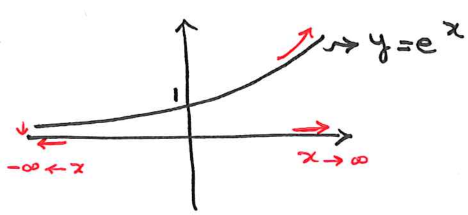
Note that "a" can be a finite number or $\pm\infty$.

Example 1. $\lim\limits_{x \to \infty} \frac{e^x}{x} = \frac{e^\infty}{\infty} = \frac{\infty}{\infty}$ !

l'Hôpital's rule $= \lim\limits_{x \to \infty} \frac{(e^x)'}{(x)'} = \lim\limits_{x \to \infty} \frac{e^x}{1}$

$= e^\infty = \infty$ ✓

* Note that $\lim\limits_{x \to \infty} e^x = \infty$




and $\lim\limits_{x \to -\infty} e^x = 0$

(*) Note that getting $\infty$ or $-\infty$ for a limit is OK and it's NOT indeterminate. The indeterminate forms are $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

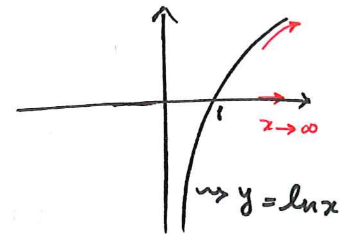
**Example 2.** $\lim_{x \to \infty} \dfrac{\ln(x)}{x} = \dfrac{\ln(\infty)}{\infty} = \dfrac{\infty}{\infty}$ !

l'Hôpital's rule

$$= \lim_{x \to \infty} \frac{(\ln(x))'}{(x)'} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x \to \infty} \frac{1}{x} = 0 \checkmark$$

⊛ Note:




$\lim_{x \to \infty} \ln x = \infty$

**Example 3.** $\lim_{x \to \infty} \dfrac{3x+7}{-5x+1} = \dfrac{3(\infty)+7}{-5(\infty)+1} = \dfrac{\infty}{\infty}$ !

a) Find this limit with l'Hôpital's rule.

b) Find this limit without l'Hôpital.

a) $\lim_{x \to \infty} \dfrac{3x+7}{-5x+1} = \lim_{x \to \infty} \dfrac{(3x+7)'}{(-5x+1)'} = \dfrac{3}{-5} \checkmark$

b) Without l'Hôpital, we should factor the dominant term out.

$$\lim_{x \to \infty} \frac{\overset{\text{dominant}}{3x}+7}{\underset{\text{dominant}}{-5x}+1} = \lim_{x \to \infty} \frac{3\cancel{x}\left(\frac{3x}{3x} + \frac{7}{3x}\right)}{-5\cancel{x}\left(\frac{-5x}{-5x} + \frac{1}{-5x}\right)}$$

$$= \lim_{x \to \infty} \frac{3\left(1 + \cancel{\tfrac{7}{3x}}\right)^{\to 0}}{-5\left(1 - \cancel{\tfrac{1}{5x}}\right)^{\to 0}} \quad \text{as } x \to \infty$$

$$= -\frac{3}{5} \checkmark$$