

**MATH 110 Midterm 2 Exam, February 27th, 2018****Duration: 90 minutes***This test has 7 questions on 9 pages, for a total of 53 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	6	7	Total
Points:	12	6	8	8	8	6	5	53
Score:								

**Student Conduct during Examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - speaking or communicating with other examination candidates, unless otherwise authorized;
  - purposely exposing written papers to the view of other examination candidates or imaging devices;
  - purposely viewing the written papers of other examination candidates;
  - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

**Full-Solution Problems.** In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are required**.

12 marks

1. (a) Find  $\frac{dy}{dx}$  if  $x^2y + y^4 = 4 + 2x$ .

**Solution:**

$$2xy + x^2y' + 4y^3y' = 2$$

$$(x^2 + 4y^3)y' = 2 - 2xy$$

$$y' = \frac{2 - 2xy}{x^2 + 4y^3}$$

- (b) Find all vertical asymptotes of  $g(x) = \frac{x^2 - 4}{x^2 + 5x + 6}$ .

**Solution:** Since we have a rational function, the candidate points for a vertical asymptote are the roots of the denominator.

$$x^2 + 5x + 6 = 0$$

$$(x + 3)(x + 2) = 0$$

So  $x = -3$  and  $x = -2$ .

To check whether these are indeed vertical asymptotes, we take the limit as  $x \rightarrow -3$  and  $x \rightarrow -2$ .

$$\lim_{x \rightarrow -3} \frac{x^2 - 4}{x^2 + 5x + 6} = \lim_{x \rightarrow -3} \frac{(x - 2)(x + 2)}{(x + 3)(x + 2)} = \lim_{x \rightarrow -3} \frac{(x - 2)}{(x + 3)}$$

Then we check the left and right limits

$$\lim_{x \rightarrow -3^+} \frac{(x - 2)}{(x + 3)} = \frac{-5}{0^+} = -\infty$$

$$\lim_{x \rightarrow -3^-} \frac{(x - 2)}{(x + 3)} = \frac{-5}{0^-} = +\infty$$

Since these limits are infinite, we conclude that  $x = -3$  is a vertical asymptote.

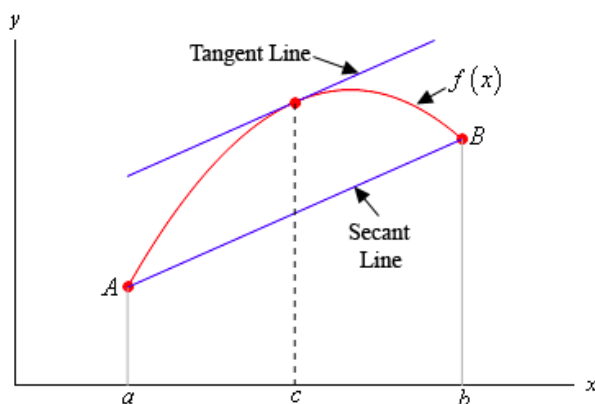
Next, we evaluate

$$\lim_{x \rightarrow -2} \frac{x^2 - 4}{x^2 + 5x + 6} = \lim_{x \rightarrow -2} \frac{(x - 2)(x + 2)}{(x + 3)(x + 2)} = \lim_{x \rightarrow -2} \frac{(x - 2)}{(x + 3)} = \frac{-4}{+1} = -4$$

Since the limit is finite, we conclude that  $x = -2$  is not a vertical asymptote.

- (c) Sketch the graph of a function that satisfies the assumptions (or hypotheses) of the Mean Value Theorem on an interval  $[a, b]$ . On your sketch also show clearly the conclusion of the Mean Value Theorem.

**Solution:** For any function which is continuous on the interval  $[a, b]$  and differentiable on  $(a, b)$ , there is a number  $c$  between  $a$  and  $b$  such that the secant line through the endpoints is parallel to the tangent line at  $c$ . One possible graph is the following



- (d) Consider the function  $f(x) = xe^{-ax}$ , where  $a$  is an unknown constant. Determine the value of  $a$  so that  $f$  has a critical number at  $x = 2$ .

**Solution:**

$$f'(x) = e^{-ax} + xe^{-ax}(-a)$$

The derivative is defined for all  $x$ , so if  $x = 2$  is a critical number then  $f'(2) = 0$ . Thus,

$$f'(2) = e^{-2a} - 2ae^{-2a} = 0$$

$$e^{-2a}(1 - 2a) = 0$$

$$a = \frac{1}{2}$$

6 marks

2. The equation

$$e^{-2x} - \sin y = y^2 + 1$$

describes points  $(x, y)$  on a curve in the  $xy$ -plane. Find  $\frac{dy}{dx}$ .

**Solution:**

$$-2e^{-2x} - (\cos y)y' = 2yy'$$

$$(-2y - \cos y)y' = 2e^{-2x}$$

$$y' = \frac{-2e^{-2x}}{2y + \cos y}$$

Find the equation of the tangent line at the point  $(0, 0)$ , if it exists.

**Solution:** The slope of the tangent line at  $(0, 0)$  is

$$m = \frac{-2e^{-2 \cdot 0}}{2 \cdot 0 + \cos(0)} = \frac{-2}{1} = -2$$

So the line has equation  $y = -2x$ .

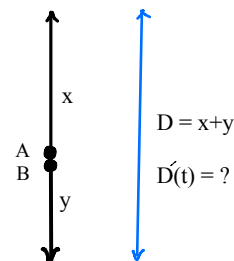
8 marks

3. Two cyclists are on a north-south straight road. They both start from the same point on the road. Cyclist A rides north at a rate of 2 m/sec and 7 seconds later cyclist B starts riding south at 1 m/sec. At what rate is the distance separating the two cyclist changing 25 seconds after cyclist A starts riding her bike?

**Solution:**

The first step is drawing a diagram. The distance of the two cyclists from the point of start increases, so  $x'(t) = 2, y'(t) = 1$  m/sec are both positive. Label the distance between the two cyclists  $D$ , in this case we have

$$D = x + y \Rightarrow D'(t) = x'(t) + y'(t) = 2 + 1 = 3 \text{ m/sec}$$



Now suppose each cyclist is on a different road. Both roads are parallel, straight and in the north-south direction and 68 metres apart. Both cyclists start at similar points on each road (that is, the cyclists are 68 metres apart at the start). Cyclist A rides north at a rate of 2 m/sec and 7 seconds later cyclist B starts riding south at 1 m/sec. At what rate is the distance separating the two cyclist changing 25 seconds after cyclist A starts riding her bike?

**Solution:**

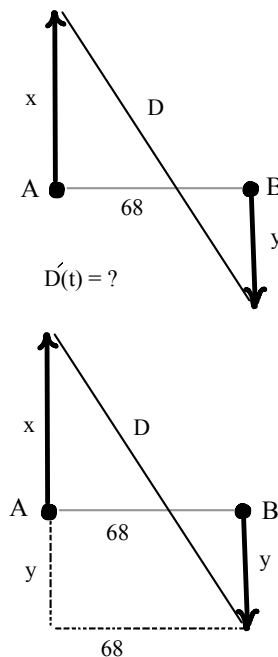
This time the two cyclists are travelling on different roads, as given in the diagram on the right. We can make a right triangle for which  $D'(t)$  is the unknown. The length of one leg of this triangle is fixed and equal to 68 m, the length of the other leg is the sum of the distances that each cyclist travels. By Pythagorean formula we have

$$D^2 = 68^2 + (x + y)^2 \Rightarrow 2DD' = 0 + 2(x + y)(x' + y')$$

First, we should find  $x, y$  and  $D$ . Cyclist A after 25 seconds has travelled  $x(25) = 2 \times 25 = 50$  m and since cyclist B started 7 seconds later, the time for cyclist B is  $25 - 7 = 18$  seconds, and the distance  $y$  at this moment is  $y(18) = 1 \times 18 = 18$  m, so

$$x + y = 50 + 18 = 68, \quad D = \sqrt{68^2 + 68^2} = 68\sqrt{2}$$

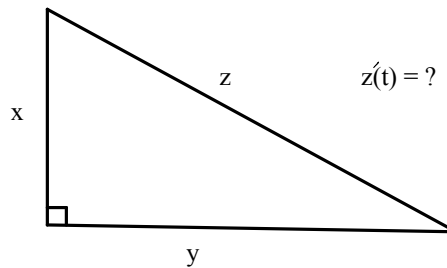
$$\text{and } D' = \frac{(x + y)(x' + y')}{D} = \frac{68 \times 3}{68\sqrt{2}} = \frac{3}{\sqrt{2}} \text{ m/sec.}$$



8 marks

4. In a right triangle, the lengths of all sides are changing in such a way that the area of the triangle remains constant and is always equal to  $6 \text{ m}^2$ . Suppose  $x$  and  $y$  are the two legs (that is, the two sides that meet at a right angle) and  $z$  is the hypotenuse and  $x$  is increasing at the rate of  $2 \text{ m/s}$ . How fast is the hypotenuse changing when  $x = 3 \text{ m}$ ?

**Solution:** The first step is to draw a diagram and label the variables as instructed.



We know that the area of this right triangle is  $A = 6 = \frac{1}{2}xy$  and also by Pythagorean formula we have  $z^2 = x^2 + y^2$ . The unknown is  $z'(t)$ , for which we need to differentiate both sides  $2zz' = 2xx' + 2yy'$  and dividing by 2 gives

$$zz' = xx' + yy'$$

The given information is  $x = 3 \text{ m}$  and  $x'(t) = 2 \text{ m/s}$ . We need to find  $y, y'$  and  $z$  first. To find  $y$  and  $y'$  we use the area formula and its derivative:

$$A = \frac{1}{2}xy \Rightarrow 6 = \frac{1}{2}3y \Rightarrow y = 4$$

$$\text{differentiate} \Rightarrow 0 = \frac{1}{2}(x'y + xy') = \frac{1}{2}(2 \times 4 + 3 \times y') \Rightarrow y' = -\frac{8}{3} \text{ m/s}$$

To find  $z$ , we use Pythagorean formula

$$z = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Now we are ready to find  $z'$ :

$$z' = \frac{xx' + yy'}{z} = \frac{(3 \times 2) + (4 \times -\frac{8}{3})}{5} = -\frac{14}{15} \text{ m/s}$$

8 marks

5. Let  $f(t) = t^4 - 8t^3 + 17$ .(a) Find the intervals on which  $f(t)$  is increasing or decreasing.**Solution:**

$$f'(t) = 4t^3 - 24t^2 = 4t^2(t - 6)$$

$f'(t)$  is defined for all  $t$  and  $f'(t) = 0$  at  $t = 0$  and  $t = 6$ . So the critical numbers are  $t = 0$  and  $t = 6$ . By testing points we find that  $f'(t) > 0$  for  $t > 6$  and  $f'(t) < 0$  for  $t < 0$  and  $0 < t < 6$ . So  $f$  is increasing for  $t > 6$  and decreasing for  $t < 0$  and  $0 < t < 6$ .

(b) Find the intervals on which  $f(t)$  is concave up or down.**Solution:**

$$f''(t) = 12t^2 - 48t = 12t(t - 4)$$

$f''(t)$  is defined for all  $t$  and  $f''(t) = 0$  at  $t = 0$  and  $t = 4$ . By testing points we find that  $f''(t) > 0$  for  $t < 0$  and  $t > 4$  and  $f''(t) < 0$  for  $0 < t < 4$ . Thus  $f$  is concave up for  $t < 0$  and  $t > 4$  and concave down for  $0 < t < 4$ .

(c) The point  $(0, 17)$  is

(i) a local maximum.

(ii) a local minimum.

**(iii) an inflection point.**

(iv) none of the above.

(d) Compute the following limits:

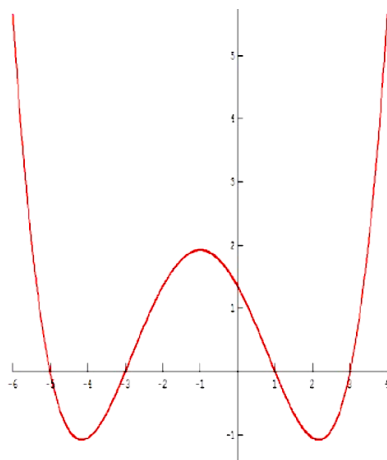
$$\lim_{t \rightarrow \infty} f(t)$$

$$\lim_{t \rightarrow -\infty} f(t)$$

**Solution:** Since  $f$  is a polynomial of fourth degree,  $\lim_{t \rightarrow \infty} f(t) = +\infty$  and  $\lim_{t \rightarrow -\infty} f(t) = +\infty$ .

6 marks

6. Let  $f$  be a continuous function on  $(-\infty, +\infty)$ . Below is the graph of **the first derivative**  $f'$ . Use this graph to answer the questions below. When reading the graph, round numerical values to the nearest integer. Justify your answers.



- (a) Determine the intervals in which  $f$  is increasing and the intervals in which  $f$  is decreasing.
- (b) Determine the  $x$ -coordinate of any local extreme value of  $f$ . Identify whether it corresponds to a local maximum or a local minimum of  $f$ .
- (c) Determine the intervals in which  $f$  is concave down and concave up.
- (d) Identify the  $x$ -coordinate of any inflection points.

**Solution:**

(a)  $f$  is increasing for  $x < -5$ ,  $-3 < x < 1$ ,  $x > 3$  because as showed on the given graph,  $f'(x) > 0$  on those intervals. Similarly,  $f$  is decreasing for  $-5 < x < -3$  and  $1 < x < 3$  because  $f'(x) < 0$  on those intervals.

(b) Based on the information described in (a), we find a local maximum at  $x = -5$  and  $x = 1$  and a local minimum at  $x = -3$  and  $x = 3$ .

(c) From the given graph, we see that  $f'$  is increasing for  $-4 < x < -1$  and  $x > 2$  and decreasing for  $x < -4$  and  $-1 < x < 2$ , thus  $f$  is concave up for  $-4 < x < -1$  and  $x > 2$  and concave down for  $x < -4$  and  $-1 < x < 2$ .

(d) Based on the information in part (c), we find inflection points at  $x = -4$ ,  $x = -1$ ,  $x = 2$ .



5 marks

7. Two friends, Athena and Bob, have a foot race. They each finish right around 60 seconds have passed, but the race is too close to call. Athena claims that she finished first, but Bob claims that they both finished at exactly 60 seconds.

By comparing smartwatch data, they learn that at any given moment, Athena was running faster than Bob. Upon learning this, Bob agrees that there is no way he could have tied Athena. How does Athena know he was right? Provide an argument built on mathematical concepts and theorems discussed in class. You may assume functions related to human motion are differentiable.

**Solution:** Let  $a(t)$  denote the position of Athena at time  $t$ , and let  $b(t)$  denote the position of Bob. If Bob did tie Athena, then the difference  $d(t) = a(t) - b(t)$  between their positions must have been zero at time  $t = 60$ .

If this is the case, then we can apply Rolle's theorem to the function  $d(t)$  on the interval  $[0, 60]$ , because  $d(0) = d(60) = 0$ , and deduce that there is some time  $t = c$ , for  $0 < c < 60$ , such that  $d'(t) = 0$ . But

$$d'(t) = a'(t) - b'(t),$$

and so  $d'(t) = 0$  would imply  $a'(t) = b'(t)$ . Thus, if Athena and Bob did tie, then Bob's speed must have been the same as Athena's at time  $t = c$ , which the smartwatch data shows not to be the case. So Athena must have won the race.