Solution to limit problems :

a) $\lim\limits_{x\to 0} \dfrac{\ln(1+3x)}{5x} \overset{\nearrow}{=}$ substitute first $\dfrac{\ln(1+0)}{0} = \dfrac{\ln 1}{0} = \dfrac{0}{0} \rightsquigarrow$ l'Hôpital

$$= \lim_{x\to 0} \frac{(\ln(1+3x))'}{(5x)'} = \lim_{x\to 0} \frac{\frac{1}{1+3x}\cdot 3}{5} = \frac{\frac{1}{1+0}\cdot 3}{5} = \frac{3}{5} \checkmark$$

b) $\lim\limits_{x\to\infty} (x+1)^3 - x^3 = \infty - \infty \rightsquigarrow$ indeterminate

modify $\longrightarrow \lim\limits_{x\to\infty} (\cancel{x^3} + 3x^2 + 3x + 1) - \cancel{x^3}$

dominant

$$= \lim_{x\to\infty} 3x^2 + 3x + 1$$

$$= \lim_{x\to\infty} 3x^2\left(1 + \frac{3x}{3x^2} + \frac{1}{3x^2}\right)$$

$= \lim\limits_{x\to\infty} 3x^2 = \infty \rightsquigarrow$ NOT indeterminate anymore ✓

c) $\lim\limits_{x\to\infty} e^{-x} + x^{-1} = \lim\limits_{x\to\infty} \dfrac{1}{e^x} + \dfrac{1}{x} = \dfrac{1}{e^\infty} + \dfrac{1}{\infty}$

write with positive powers

$= 0 + 0 = 0$ ✓

You can also find the common denominator :

$\lim\limits_{x\to\infty} \dfrac{x + e^x}{xe^x} = \dfrac{\infty}{\infty}$ and if you apply l'Hôpital's rule twice you will get the same result, 0 .

d) $\displaystyle\lim_{x\to 1} \frac{2}{x-1} - \frac{1}{\underbrace{x^2-1}_{(x-1)(x+1)}} = \frac{2}{1-1} - \frac{1}{1-1} = \frac{2}{0} - \frac{1}{0} = \infty - \infty$

indeterminate

common denominator

$$= \lim_{x\to 1} \frac{2(x+1)-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{2x+1}{(x-1)(x+1)} = \frac{2+1}{(1-1)(1+1)} = \frac{3}{0}$$

So $\displaystyle\lim_{x\to 1^+} f(x) = \frac{3}{0^+} = +\infty$ ✓

and $\displaystyle\lim_{x\to 1^-} f(x) = \frac{3}{0^-} = -\infty$ ✓

e) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln(x) = \sqrt{0} \times \ln(0) = 0 \times \infty$ ⇝ Rewrite $\sqrt{x}\ln(x)$ as a fraction.

trick

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{\sqrt{x}} \to x^{-\frac{1}{2}}} = \frac{\ln(0)}{\frac{1}{0}} = \frac{\infty}{\infty}$$ ⇝ Now l'Hôpital's rule

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \lim_{x\to 0^+} -2\,\frac{\frac{1}{x}}{\frac{1}{x^{\frac{3}{2}}}}$$

algebra to simplify

$$= \lim_{x\to 0^+} -2\,\frac{x^{\frac{3}{2}}}{x}$$

$$= \lim_{x\to 0^+} -2x^{\frac{1}{2}} = \lim_{x\to 0^+} -2\sqrt{x}$$

$$= -2\sqrt{0} = 0 \checkmark$$

f) $\displaystyle\lim_{x\to\infty} x^3 e^{-3x^2} =$

negative power

$$= \lim_{x\to\infty} \frac{x^3}{e^{3x^2}} = \frac{\infty}{e^{\infty}} = \frac{\infty}{\infty}$$ ⇝ l'Hôpital

$$= \lim_{x\to\infty} \frac{3x^{\cancel{2}}}{\underset{2}{\cancel{6}}\cdot\cancel{x}\, e^{3x^2}} = \lim_{x\to\infty} \frac{x}{2e^{3x^2}} = \frac{\infty}{\infty}$$ ⇝ Once again l'Hôpital

$$= \lim_{x\to\infty} \frac{1}{2\times 6x\, e^{3x^2}} = \lim_{x\to\infty} \frac{1}{12x\, e^{3x^2}} = \frac{1}{\infty} = 0 \checkmark$$

g) $\lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right) \rightsquigarrow$ Note : when $x \to \infty$, $\frac{1}{x} \to 0$

so $e^{\frac{1}{x}} \to e^0 = 1$

Therefore, $x\left(e^{\frac{1}{x}} - 1\right) \longrightarrow \infty(1-1) = \infty \times 0$ when $x \to \infty$

So we need to write it as a fraction:

trick $\lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \underset{\text{substitute}}{=} \frac{e^{\frac{1}{\infty}} - 1}{\frac{1}{\infty}} = \frac{e^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$!

l'Hôpital

$$= \lim_{x\to\infty} \frac{\left(e^{\frac{1}{x}} - 1\right)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to\infty} \frac{\cancel{-\frac{1}{x^2}}\left(e^{\frac{1}{x}} - 1\right)}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to\infty} e^{\frac{1}{x}} - 1 = 0 \checkmark$$

h) $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{e^\infty} = \frac{\infty}{\infty} \rightsquigarrow$ l'Hôpital.

negative power

$$= \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$$

one more l'Hôpital

$$= \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0 \checkmark$$

i) $\lim\limits_{x \to 2^+} e^{\frac{1}{2-x}} \overset{\text{substitute}}{=} e^{\frac{1}{2-2^+}} = e^{\frac{1}{0^-}} = e^{-\infty} = 0$ ✓

Note that : $\lim\limits_{\substack{x \to 2^+ \\ \downarrow \\ 2.1}} \frac{1}{2-x} = \frac{1}{2-2.1} = \frac{1}{0^-} = -\infty$

So $\lim\limits_{x \to 2^+} e^{\frac{1}{2-x}} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$ ✓

j) $\lim\limits_{x \to 0} \frac{1}{x} - \frac{1}{\sin x} = \frac{1}{0} - \frac{1}{\sin 0} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$

Common denominator

$$= \lim_{x \to 0} \frac{\sin x - x}{x \sin x} = \frac{\sin 0 - 0}{0 \sin 0} = \frac{0}{0} \rightsquigarrow \text{l'Hôpital}$$

l'Hôpital

$$= \lim_{x \to 0} \frac{\cos x - 1}{\underbrace{\sin x + x \cos x}_{\text{product rule for the bottom}}} \overset{\text{sub again}}{=} \frac{\cos 0 - 1}{\sin 0 + 0 \cos 0} = \frac{0}{0}$$

One more l'Hôpital

$$= \lim_{x \to 0} \frac{-\sin x}{\cos x + \cos x - x \sin x} \overset{\text{sub}}{=} \frac{-\sin 0}{\cos 0 + \cos 0 - 0 \sin 0}$$

$$= \frac{0}{2 - 0} = \frac{0}{2} = 0 \checkmark$$

Some Remarks :

- The first step in finding a limit is always direct substitution, there are many cases that no l'Hôpital's rule is required.
- l'Hôpital's rule can be applied to only indeterminate limits in the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$. If you apply l'Hôpital incorrectly, you'll get a wrong number for the limit.
- Sometimes algebraic manipulation of the function helps us get rid of $0 \times \infty$ or $\infty - \infty$ or other bad cases.