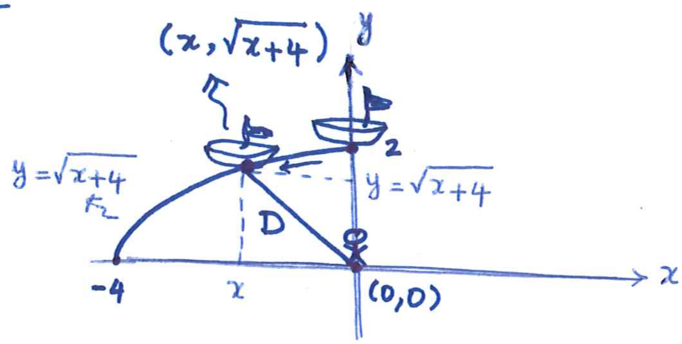
Solution HW5

Q1



The distance between you and the boat is

$$D = \sqrt{(x-0)^2 + (\sqrt{x+4}-0)^2}$$

$$\Rightarrow \quad D = \sqrt{x^2+x+4}\ , \quad -4 \leq x \leq 0$$

There are two ways to minimize D:

(1) Work directly with

$$D(x) = \sqrt{x^2+x+4}\ , \quad -4 \leq x \leq 0$$

$$D(x) = (x^2+x+4)^{\frac{1}{2}}$$

$$D'(x) = (2x+1)\left(\tfrac{1}{2}\right)(x^2+x+4)^{-\frac{1}{2}}$$

$$D'(x) = \frac{2x+1}{2\sqrt{x^2+x+4}}$$

$$D'(x) = 0 \Rightarrow 2x+1 = 0 \Rightarrow x = -\frac{1}{2}$$

* Note that the denominator is always positive, so no need to worry about $D'(x)$ undefined.

- Is $x = -\frac{1}{2}$ a local minimum?

1st derivative test is easier:

x	-4	$-\frac{1}{2}$	0
$D'(x)$	$-$		$+$
$D(x)$	$\searrow$	local min	$\nearrow$

(2) Work with D^2: x^2+x+4

take

$$f(x) = x^2+x+4$$

and use the point that the critical numbers and extrema of $f(x)$ and $\sqrt{f(x)}$ for any function $f(x) \geq 0$ are the same. (This is done in a workshop.)

$$f'(x) = 2x+1$$

$$f'(x) = 0 \Rightarrow 2x+1 = 0$$

$$\Rightarrow x = -\frac{1}{2}$$

- Is it a local min?

2nd derivative test is easier:

$f''(x) = 2 \rightarrow$ always positive $\downarrow$ concave up

$f''(-\frac{1}{2}) > 0$

local min

• Is it a global minimum ?

Check the endpoints :

$D(x) = \sqrt{x^2+x+4} \quad -4 \leq x \leq 0$

$D(0) = \sqrt{4} = 2$

$D(-4) = \sqrt{16-4+4} = \sqrt{16} = 4$

$D(-\frac{1}{2}) = \sqrt{\frac{1}{4} - \frac{1}{2} + 4} = \sqrt{-\frac{1}{4}+4} = \sqrt{\frac{15}{4}}$

$= \frac{\sqrt{15}}{2}$

↓

smallest

So $D(x)$ has its global min at $x = -\frac{1}{2}$ and $D(-\frac{1}{2}) = \frac{\sqrt{15}}{2}$.

• Is it a global min ?

$-4 \leq x \leq 0$

$f(-4) = 16-4+4 = 16$

$f(0) = 4$

$f(-\frac{1}{2}) = \frac{15}{4}$ ⇝ smallest

So $f(x)$ has a global min at $x = -\frac{1}{2}$, y-coordinate : $\sqrt{-\frac{1}{2}+4} = \sqrt{\frac{7}{2}}$

and

closest distance :

$\sqrt{f(x)} = \sqrt{f(-\frac{1}{2})} = \sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$

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y - coordinate : $y = \sqrt{x+4} = \sqrt{-\frac{1}{2}+4} = \frac{\sqrt{7}}{\sqrt{2}}$

Point : $(-\frac{1}{2}, \sqrt{\frac{7}{2}})$, closest distance $= \frac{\sqrt{15}}{2}$

Q2. $Q(p) = 6 - 2\sqrt{p}$, $R(p) =$ price . quantity

$\Rightarrow R(p) = p \cdot Q(p) = p(6-2\sqrt{p})$

We want to find the global max for

$R(p) = 6p - 2p\sqrt{p}$

Note that $p \geq 0$. but also Revenue can NOT be negative

So solve : $R(p) = 0 \quad 2p(3-\sqrt{p}) = 0 \rightarrow p = 0$

$\rightarrow 3 = \sqrt{p} \Rightarrow p = 9$

So the bounds on p are : $0 \leq p \leq 9$

(If $p > 9$ then we get $Q(p) < 0 \rightsquigarrow$ Negative quantity !)

So maximize $R(p) = 6p - 2p^{3/2}$ on the interval $[0, 9]$

$$R'(p) = 6 - 2 \cdot \frac{3}{2} p^{\frac{1}{2}} = 6 - 3\sqrt{p}$$

$$R'(p) = 0 \Rightarrow 6 - 3\sqrt{p} = 0 \Rightarrow 6 = 3\sqrt{p}$$

$$\Rightarrow 2 = \sqrt{p}$$

$$\Rightarrow p = 4$$

- Is it a local max ?

2^{nd} derivative test : $R''(p) = -\frac{3}{2} p^{-\frac{1}{2}}$

$$\Rightarrow R''(4) = -\frac{3}{2} \frac{1}{\sqrt{4}} < 0$$

local max

- Is it a global max ?

Check the endpoints: $0 \leq p \leq 9$

$R(0) = 0$

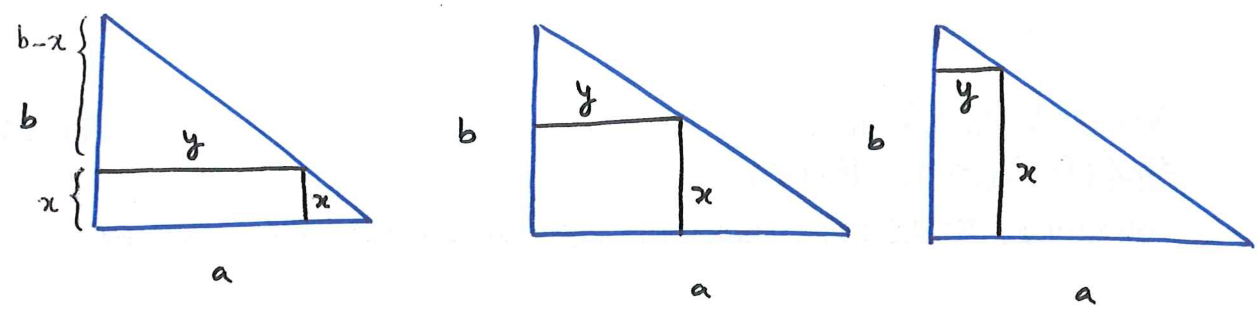
$R(9) = 6 \cdot 9 - 2 \cdot 9\sqrt{9} = 54 - 54 = 0$

$R(4) = 6 \cdot 4 - 2 \cdot 4\sqrt{4} = 24 - 16 = 8 \rightsquigarrow$ largest

$\Rightarrow$ Maximum Revenue is when $p = 4000\$$

and revenue becomes $R(4) = 8000\$$.

Q3.



We are given a fixed-size triangle with legs "a" and "b". We want to find the rectangle with maximum area. What should x and y be?

Objective function: $A = xy$

Constraint: inside the right triangle

Use similar triangles to find x or y in terms of the other

$$\frac{y}{\underset{\text{Constant}}{a}} = \frac{\overset{\text{Constant}}{b-x}}{\underset{\text{constant}}{b}}$$

$$\Rightarrow yb = a(b-x)$$

$$\Rightarrow y = \frac{a(b-x)}{b}$$

Rewrite the objective function:

$$A = xy = x \cdot \frac{a(b-x)}{b}$$

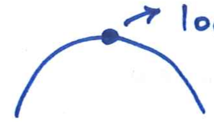
Simplify $A(x) = \frac{a}{b}(bx - x^2)$ Domain: $0 \le x \le b$

We want to maximize $A(x)$ on the interval $[0,b]$.

$$A'(x) = \frac{a}{b}(b - 2x) = 0 \Rightarrow b - 2x = 0 \Rightarrow x = \frac{b}{2}$$

- Is $x = \frac{b}{2}$ a local max?

2^{nd} derivative test: $A''(x) = \frac{a}{b}(-2) < 0 \rightsquigarrow$ always concave down



→ local max.

- Is $x = \frac{b}{2}$ a global max?

Check the values at the endpoints and at $x = \frac{b}{2}$:

$A(x) = \frac{a}{b}(bx - x^2)$; $\quad 0 \le x \le b$

$A(0) = 0$

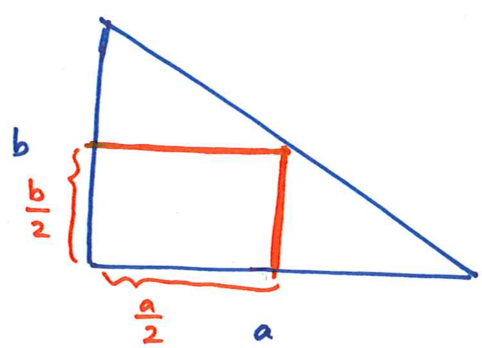
$A(b) = \frac{a}{b}(b \cdot b - b^2) = 0$

$A\left(\frac{b}{2}\right) = \frac{a}{b}\left(b \cdot \frac{b}{2} - \left(\frac{b}{2}\right)^2\right) = \frac{a}{b}\left(\frac{b^2}{2} - \frac{b^2}{4}\right) = \frac{a}{b} \cdot \frac{b^2}{4} = \frac{ab}{4} \rightsquigarrow$ largest

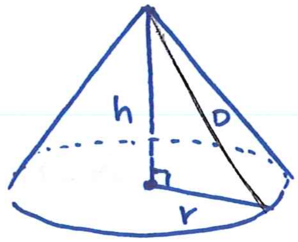
So the maximum area is when:

$x = \frac{b}{2}$ and $\quad y = \frac{a}{b}(b - x) = \frac{a}{b}\left(b - \frac{b}{2}\right) = \frac{a}{b} \cdot \frac{b}{2} = \frac{a}{2}$

So $A = xy = \frac{b}{2} \cdot \frac{a}{2} = \frac{ab}{2}$



Q4.



Objective function:

$$V = \frac{1}{3}\pi r^2 \cdot h$$

Constraint: Right triangle

$\Downarrow$

Use Pythagorean to write r or h in terms of the other

$$h^2 + r^2 = D^2 \quad \text{constant}$$

$$r^2 = D^2 - h^2$$

(*) Note: It's easier to write r in terms of h.
We'd like to avoid working with: $h = \sqrt{D^2 - r^2}$

Rewrite the objective function: $V = \frac{1}{3}\pi(D^2 - h^2) \cdot h$

$$\Rightarrow \quad V(h) = \frac{\pi}{3}(D^2 h - h^3) \qquad \text{Domain: } h > 0$$

Maximize V:

$$V'(h) = \frac{\pi}{3}(D^2 - 3h^2) = 0 \Rightarrow D^2 - 3h^2 = 0$$

$$\Rightarrow D^2 = 3h^2$$

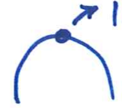
$$\Rightarrow h^2 = \frac{D^2}{3}$$

$$\Rightarrow h = \pm\sqrt{\frac{D^2}{3}} \Rightarrow = \frac{D}{\sqrt{3}}$$

- Is $h = \frac{D}{\sqrt{3}}$ a local max?

2nd derivative test :

$$V''(h) = \frac{\pi}{3}(-6h) \xrightarrow{h = \frac{D}{\sqrt{3}}} V''\left(\frac{D}{\sqrt{3}}\right) < 0 \rightsquigarrow$$



local max.

- Is $h = \frac{D}{\sqrt{3}}$ a global max?

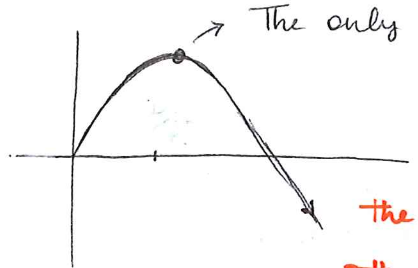
Visualize the graph of V.

$$V(h) = \frac{\pi}{3}\left(D^2 h - h^3\right)$$

has a local max at $h = \frac{D}{\sqrt{3}}$

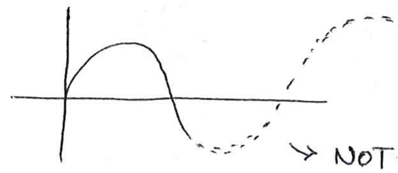
and it has NO other critical number.

So the function for $h > 0$ should be always concave down.



The only local max.

the function should go down. Otherwise it should have another critical number which contradicts our computation.



NOT possible to have another global max

Therefore ; the volume of the cone is maximized when $h = \frac{D}{\sqrt{3}}$ and

$$r = \sqrt{D^2 - h^2} = \sqrt{D^2 - \left(\frac{D}{\sqrt{3}}\right)^2} = \sqrt{D^2 - \frac{D^2}{3}} = \sqrt{\frac{2D^2}{3}} = \sqrt{\frac{2}{3}}\, D$$

and Volume is :

$$V = \frac{1}{3}\pi \cdot \frac{2}{3} D^2 \cdot \frac{D}{\sqrt{3}} = \frac{2\pi}{9\sqrt{3}} D^3$$