For students who will take the course <u>Integral Calculus</u>.

You need to know the following topics:

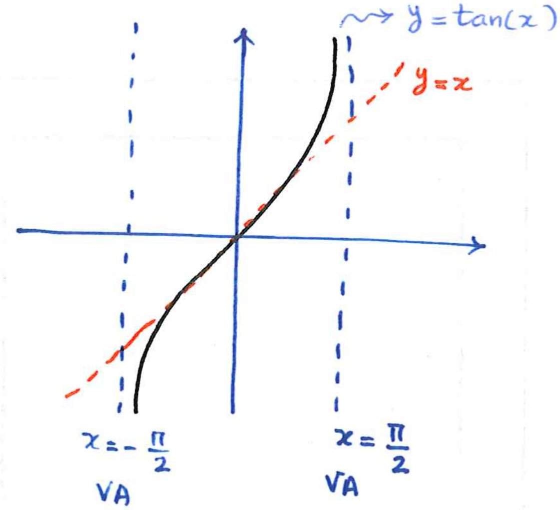
(1) Inverse of the function $y = \tan(x)$ is denoted by

$$y = \arctan(x) \quad \text{or} \quad \operatorname{Arctan}(x) \quad \text{or} \quad \tan^{-1}(x)$$

Recall the shape of $y = \tan(x)$

Domain : $(-\frac{\pi}{2}, \frac{\pi}{2})$

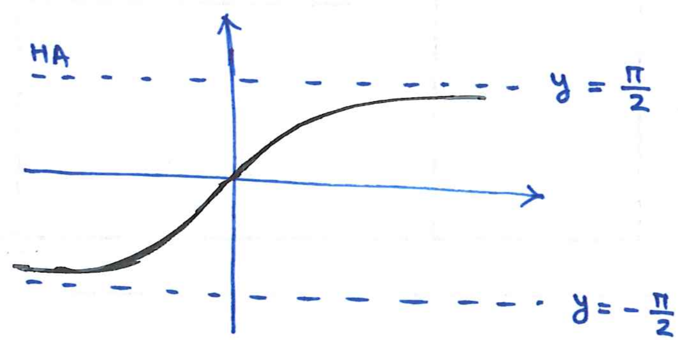
Range : $\mathbb{R}$



To find the graph of $y = \arctan(x)$ as we learned in term 1 we should reflect the above graph across the diagonal line $y = x$, then you'll have

Domain : $\mathbb{R}$

Range : $(-\frac{\pi}{2}, \frac{\pi}{2})$



Derivative of $\arctan(x)$:

$$y = \arctan(x) \quad \Rightarrow \quad y' = \frac{1}{1+x^2}$$

$$y = \arctan(f(x)) \xrightarrow[\text{rule}]{\text{chain}} y' = \frac{1}{1+(f(x))^2} \cdot f'(x)$$

(2) Functions of the form $y = x^x$ or in general $y = (f(x))^{g(x)}$

To deal with these type of functions, we need to use the following properties of the functions e^x and $\ln(x)$.

$$\begin{cases} 1.\ e^{\ln(x)} = x, \quad e^{\ln(f(x))} = f(x) \\ 2.\ \ln x^n = n \cdot \ln(x), \quad \ln(f(x))^n = n \cdot \ln(f(x)) \end{cases}$$

Now, let's find the following limit:

$$\lim_{x \to 0} x^x = 0^0 \rightsquigarrow \text{indeterminate limit.}$$

How to resolve this? Use the two tricks above:

$$x^x \underset{\text{property (1)}}{=} e^{\ln(x^x)} \underset{\text{property 2.}}{=} e^{x \ln x}$$

So $\displaystyle \lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = e^{0 \cdot \ln(0)} = e^{0 \cdot \infty}$ ← still indeterminate

How to resolve $0 \cdot \infty$? Use the trick $a \cdot b = \dfrac{a}{\frac{1}{b}}$ or $\dfrac{b}{\frac{1}{a}}$

$$\lim_{x \to 0} x^x = \lim_{x \to 0} e^{x \ln x} = \lim_{x \to 0} e^{\frac{\ln x}{\frac{1}{x}}} = \lim_{x \to 0} e^{\frac{\ln 0}{\frac{1}{0}}} = \lim_{x \to 0} e^{\frac{\infty}{\infty}}$$

still indeterminate!

How to resolve this? Use l'Hôpital's rule

$$\lim_{x\to 0} e^{\frac{\ln x}{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = \lim_{x\to 0} e^{-\frac{x^2}{x}}$$

$$= \lim_{x\to 0} e^{-x} = e^0 = 1$$

So
$$\boxed{\lim_{x\to 0} x^x \underbrace{=}_{\text{log tricks}} \lim_{x\to 0} e^{x\ln x} \underbrace{=}_{0\cdot\infty \text{ trick}} \lim_{x\to 0} e^{\frac{\ln x}{\frac{1}{x}}} \underbrace{=}_{\text{l'Hôpital}} 1}$$

Now let's find the derivative of $y = x^x$.

Again use the trick: $y = x^x$

$$\xrightarrow[\text{from both sides}]{\text{take ln}} \ln(y) = \ln x^x$$

$$\xrightarrow{\text{Trick (2)}} \ln(y) = x \ln x$$

Now take the derivative of both sides $\quad (\ln y)' = (x \ln x)'$

$$\xrightarrow[\text{differentiation}]{\text{implicit}} \frac{1}{y}\cdot y' = 1\cdot \ln x + x\cdot\frac{1}{x}$$

$$\xrightarrow[y']{\text{solve for}} y' = y\cdot(\ln x + 1)$$

$$\xrightarrow[y = x^x]{\text{replace}} \boxed{y' = x^x(\ln x + 1)}$$