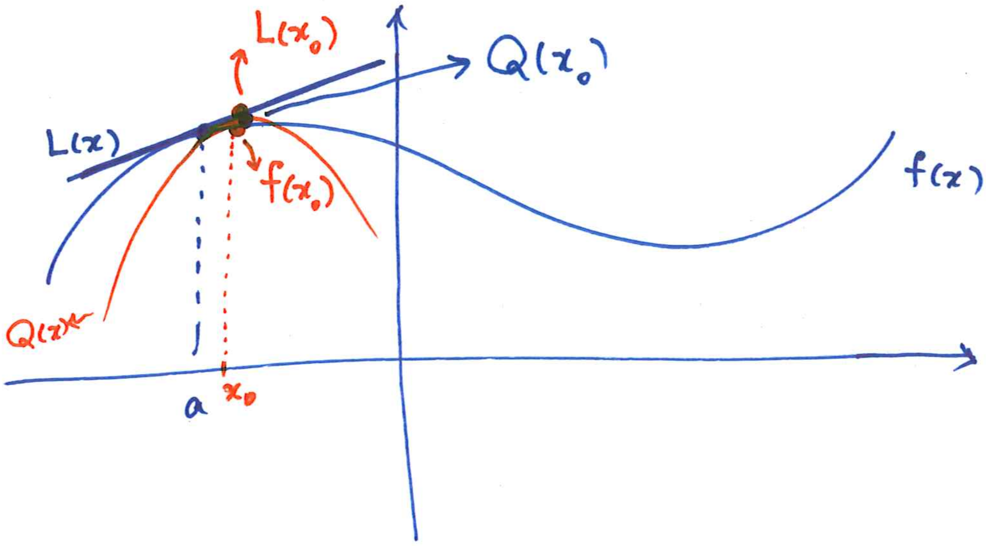
April 4

Lecture 34

Last Week: Approximating a function near to a given point.



$f(x_0) \approx L(x_0)$

$T_1(x) = \boxed{L(x) = f(a) + f'(a)(x-a)} \longrightarrow$ Linear Approx.

$T_2(x) = \boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2} \rightarrow$ Quadrat
Approx

Today: Higher degree approximations

(3rd degree, 4th degree, 5th degree .... polynomia

$$T_3(x) = \underbrace{\underbrace{f(a) + f'(a)(x-a)}_{\text{linear part}} + \frac{f''(a)}{2}(x-a)^2}_{\text{Quadratic part}} + \frac{f'''(a)}{3 \underset{\text{times}}{\times} 2}(x-a)^3$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{Cubic/ 3rd degree polynomial}}$$

$$T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \underset{\text{times}}{\times} 2}(x-a)^3 + \frac{f^{(4)}(a)}{4 \underset{\text{times}}{\times} 3 \underset{\text{times}}{\times} 2}(x-a$$

.

.

.

$T_7(x) =$ " " " " " " "

$$+ \frac{f^{(7)}(a)}{7.6.5.4.3.2.1}(x-a)^7$$

last term

Math Notation:

4th derivative :	$f^{(4)}(a)$
5th " :	$f^{(5)}(a)$
6th " :	$f^{(6)}(a)$
⋮	
nth derivative :	$f^{(n)}(a)$

Factorial Notation:

read it : 4 fact

$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

7 facto

$7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$

$5! = \underbrace{\cancel{5 \cdot 4}}_{20} \cdot \underbrace{\cancel{3 \cdot 2}}_{6} \cdot 1 = 120$

General formula for approximation a function with an n-th degree polynomial.

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Taylor Polynomial of degree "n"

Example 1. Consider the function $f(x) = \ln(1+x)$

a) Find the local linearization of f near $x=0$

b) " $T_2(x)$ for f near $x=0$

c) " $T_3(x)$ " " " "

d) estimate $\ln(0.9)$.

$f(x) = \ln(1+x)$ $\qquad \boxed{a = 0} \rightarrow$ nice point

a) $T_1(x)$ or $L(x)$

$$T_1(x) = f(a) + f'(a)(x-a)$$

$$f'(x) = \frac{1}{1+x} \xrightarrow{a=0} f'(0) = \frac{1}{1+0} = 1$$

$$f(0) = \ln(1+0) = \ln(1) = 0$$

$$T_1(x) = \underbrace{\cancel{f(0)}}_{0} + \underbrace{\cancel{f'(0)}}_{1}(x-0) \Longrightarrow \boxed{T_1(x) = x} \rightarrow \text{linear approx}$$

b) $$T_2(x) = \underbrace{\cancel{f(0)}}_{0} + \underbrace{\cancel{f'(0)}}_{1}(x-0) + \frac{f''(0) \rightarrow -1}{2! \rightarrow 2.1=2}(x-0)^2$$

Quotient Rule : $f''(x) = \dfrac{-1}{(1+x)^2} \xrightarrow{x=0} f''(0) = \dfrac{-1}{(1+0)^2} = -1$

$$\boxed{T_2(x) = x - \frac{1}{2}x^2} \rightarrow \text{Quadratic approx.}$$

c) $$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0) \rightarrow 2}{3! \rightarrow 3.2.1}(x-$$

Quotient Rule : $f'''(x) = \dfrac{2}{(1+x)^3} \rightarrow f'''(0) = \dfrac{2}{(1+0)^3} = 2$

$$\boxed{T_3(x) = x - \frac{1}{2}x^2 + \frac{2}{6}x^3} \to \text{3rd degree Taylor polynomial}$$

d) $\ln(0.9) \overset{\substack{\text{calculator}\\ \uparrow}}{=} -0.10536051\ldots$

$f(x) = \ln(1+x)$

$\cancel{f(0.9)} = \ln(1+0.9) = \ln(1.9)$

$x = -0.1 \quad , \quad f(-0.1) = \ln(1+(-0.1)) = \ln(0.9)$

$L(-0.1) = T_1(-0.1) = -0.1$

$\ln(0.9) \approx -0.1 \rightsquigarrow$ good

Or $T_2(x) \Rightarrow T_2(-0.1) = -0.1 - \frac{1}{2}(-0.1)^2$

$= -0.1 - 0.005$

$= -0.105$

$\ln(0.9) \approx -0.105 \rightsquigarrow$ better

Or $T_3(-0.1) = -0.105 + \frac{1}{3}(-0.1)^3 = -0.10533 \approx \ln($

$\rightsquigarrow$ even better ...