April 6

Lecture 35 (Last Class :( )

Anti derivative. The derivative of a function is given, find the original function.

Example 1. If $f'(x) = 2x$, what is $f(x)$?

Find $f(x)$ whose derivative is equal to $2x$ $\rightsquigarrow$ $f(x) = x^2$

Is $f(x) = x^2$ the only function?

What about $f(x) = x^2 + 1$? or $f(x) = x^2 - 10$?

or $f(x) = x^2 - \frac{1}{2}$

All of these functions have their derivative $= 2x$.

As you see when finding the antiderivative, it's possible that there's a constant term which doesn't appear in the derivative but it's part of the original function (antiderivative).

We denote the constant term by C.

So if $f'(x) = 2x$, then $\boxed{f(x) = x^2 + C}$ → general form of the antiderivative

Example 2. Now if $f'(x) = x$, then $f(x) = ?$

if $f(x) = x^2 \rightsquigarrow f'(x) = 2x$

how to get rid of 2 ? divide the original function by 2.

So $\boxed{f(x) = \frac{x^2}{2} + C}$ → Constant term always appears in the general form.

$f'(x) = \frac{1}{2} \cdot 2x$
$= x$ ✓

Ex 3 . • $f'(x) = x^2$, $f(x) = ?$

if $f(x) = x^3 \leadsto f'(x) = \frac{1}{3} \cdot 3x^2$ divide by 3 to cancel

So $\boxed{f(x) = \frac{1}{3}x^3 + C}$

• $f'(x) = x^3$, $f(x) = ?$

$\boxed{f(x) = \frac{1}{4}x^4 + C}$

Now let's find a pattern for any power function say x^n :

$$\left\{ \begin{array}{lcl} x & \xrightarrow[\text{derivative}]{\text{anti}} & \frac{1}{2}x^2 + C \\ x^2 & \leadsto & \frac{1}{3}x^3 + C \\ x^3 & \leadsto & \frac{1}{4}x^4 + C \\ \vdots & & \\ x^n & \leadsto & \frac{1}{n+1}x^{n+1} + C \end{array} \right\}$$

So $\Longrightarrow$ if $f'(x) = x^n$ ← any number except (-1)

then

$\boxed{f(x) = \frac{1}{n+1}x^{n+1} + C}$

Power Rule for Antiderivative

memorize this rule

Let's verify this rule :

$f(x) = \underbrace{\frac{1}{n+1}}_{\text{a constant multiple like } (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)} x^{n+1} + C$ then $f'(x) = \frac{1}{\cancel{n+1}} \cdot (\cancel{n+1})x^n$

So $f'(x) = x^n$ ✓

Ex 4. Find the antiderivative of

a) $f(x) = \sqrt{x}$

b) $f(x) = 2\sqrt[3]{x} - \frac{5}{x^2} - 1$

* Note that we take $f(x)$ an independent function and we find its anti-derivative.

Let's denote the antiderivative of $f(x)$; $F(x)$.

a) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ → n

$$\xrightarrow[\text{for anti-d}]{\text{Power rule}} F(x) = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C$$

$$\Rightarrow F(x) = \frac{1}{\frac{3}{2}} x^{\frac{3}{2}} + C$$

$$\Rightarrow \boxed{F(x) = \frac{2}{3} x^{3/2} + C}$$

b) $f(x) = 2\sqrt[3]{x} - \frac{5}{x^2} - 1$

↳ Rewrite : $f(x) = 2x^{\frac{1}{3}} - 5x^{-2} - 1$

$$\xrightarrow[\text{Rule}]{\text{Power}} F(x) = 2 \cdot \frac{1}{\frac{1}{3}+1} x^{\frac{1}{3}+1} - 5 \cdot \frac{1}{-2+1} x^{-2+1} - x$$

$$\longrightarrow F(x) = 2 \cdot \frac{1}{\frac{4}{3}} x^{4/3} - 5 \cdot \frac{1}{-1} x^{-1} - x + C$$

$$\Rightarrow F(x) = 2 \cdot \frac{3}{4} x^{\frac{4}{3}} + 5x^{-1} - x + C$$

$$\Rightarrow \boxed{F(x) = \frac{3}{2} x^{4/3} + \frac{5}{x} - x + C}$$

→ As you saw, we excluded the case where $n=-1$ in x^n ?

What happens if $f'(x) = x^{-1}$? What is $f(x) = ?$

Let's apply the power rule : $f(x) = \frac{1}{-1+1} x^{-1+1} = \frac{1}{0} x^0$!!!

So we cannot apply this rule.

Now if $f'(x) = \frac{1}{x}$, what is $f(x)$?

Recall: $\left[\ln(x)\right]' = \frac{1}{x}$ So : $\boxed{f(x) = \ln(x) + C}$

Let's learn a few more rules:

$$f'(x) = e^x \Rightarrow \boxed{f(x) = e^x + C}$$

$$f'(x) = \cos x \Rightarrow \boxed{f(x) = \sin x + C}$$

$$f'(x) = \sin x \Rightarrow \boxed{f(x) = -\cos x + C}$$

Ex 5. a) Find the most general antiderivative of

$$f(x) = 3x^{10} + \frac{1}{\sqrt{x}} - 2\sin x + 3e^x + 2$$

b) Find the antiderivative of $f(x)$, $F(x)$, for the above function when $F(0) = 1$.

a) Let's rewrite $f(x)$ as a power function :

$$f(x) = 3x^{10} + x^{-\frac{1}{2}} - 2\sin x + 3e^{x} + 2$$

Apply the rules :

$$F(x) = 3 \cdot \frac{1}{10+1} x^{10+1} + \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} - 2 \cdot (-\cos x) + 3e^{x} + 2x + C$$

$$\xrightarrow{\text{Simplify}} F(x) = 3 \cdot \frac{1}{11} x^{11} + \frac{1}{\frac{1}{2}} x^{\frac{1}{2}} + 2\cos x + 3e^{x} + 2x + C$$

$$\Rightarrow \boxed{F(x) = \frac{3}{11} x^{11} + 2\sqrt{x} + 2\cos x + 3e^{x} + 2x + C}$$

General antiderivative

b) How is (b) is different from (a) ?

We're given an extra info $\underline{F(0) = 1}$ about the antideriv

We use this to find the constant term C and obtain one particular antiderivative .

$$F(0) = 1$$

Sub this in x this is y-value

$$F(0) = 1 = \frac{3}{11} 0^{11} + 2\sqrt{0} + 2\cos(0) + 3e^{0} + 2(0) + C$$

$$\Rightarrow 1 = 0 + 0 + 2 + 3 + C$$

$$\Rightarrow 1 = 5 + C$$

$$\Rightarrow \boxed{-4 = C}$$ → We found C so

$$\boxed{F(x) = \frac{3}{11}x^{11} + 2\sqrt{x} + 2\cos x + 3e^{x} + 2x - 4}$$ → particular antiderivative when $F(0) = 1$

Exercise. If $f'(x) = \frac{1}{x^2} - \cos x - 1$ and $f\left(\frac{\pi}{2}\right) = -1$.

Find $f(x)$.

$$f'(x) = x^{-2} - \cos x - 1$$

$$\rightsquigarrow f(x) = \frac{1}{-2+1}x^{-2+1} - \sin x - x + C$$

So $f(x) = -\frac{1}{x} - \sin x - x + C$

Now use $f\left(\frac{\pi}{2}\right) = 1$

$x = \frac{\pi}{2}$, $y = 1$ $\Rightarrow 1 = -\frac{1}{\frac{\pi}{2}} - \sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2} + C$

$$\Rightarrow 1 = -\frac{2}{\pi} - 1 - \frac{\pi}{2} + C$$

$$\Rightarrow 1 + \frac{2}{\pi} + \frac{\pi}{2} + 1 = C$$

So

$$\boxed{f(x) = -\frac{1}{x} - \sin x - x + 2 + \frac{2}{\pi} + \frac{\pi}{2}}$$