Solution to selected problems from the final review note.

l'Hôpital's Rule

c) $\lim_{x\to\infty} \dfrac{e^{2x}+x^2}{e^x+4x} = \dfrac{e^{\infty}+\infty}{e^{\infty}+\infty} = \dfrac{\infty}{\infty}$ ⇝ indeterminate → Good for l'Hôp

always substitute first

$= \lim_{x\to\infty} \dfrac{2e^{2x}+2x}{e^x+4} \overset{\text{again}}{=} \dfrac{e^{\infty}+\infty}{\infty+4} = \dfrac{\infty}{\infty}$ ⇝ Again l'Hôp

$= \lim_{x\to\infty} \dfrac{4e^{2x}}{e^x} = \lim_{x\to\infty} \dfrac{4e^x \cdot e^x}{e^x} = \lim_{x\to\infty} 4e^x \boxed{= \infty}$

manipulate algebraically

NOT indeterminate.

f) $\lim_{x\to-\infty} xe^x = -\infty \cdot e^{-\infty} = -\infty \cdot 0$ → First rewrite the function as a fraction:

trick

$\lim_{x\to-\infty} \dfrac{e^x}{\frac{1}{x}} = \dfrac{e^{-\infty}}{\frac{1}{-\infty}} = \dfrac{0}{0}$ ⇝ Good for l'Hôp

$= \lim_{x\to-\infty} \dfrac{e^x}{-\frac{1}{x^2}} = \dfrac{e^{-\infty}}{-\frac{1}{\infty}} = \dfrac{0}{0}$! → We'll keep getting $\frac{0}{0}$, switch the top and bottom:

$\lim_{x\to-\infty} \dfrac{x}{\frac{1}{e^x}} = \dfrac{-\infty}{\frac{1}{e^{-\infty}}} = \dfrac{-\infty}{\frac{1}{0}} = \dfrac{-\infty}{\infty}$ → Good for l'Hôp

e^{-x}

$= \lim_{x\to-\infty} \dfrac{1}{-e^{-x}} = \dfrac{1}{-e^{+\infty}} = \boxed{\dfrac{1}{\infty} = 0}$ ✓

e) $\lim_{x\to1^+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right) = \dfrac{1}{1-1} - \dfrac{1}{\ln(1)} = \dfrac{1}{0} - \dfrac{1}{0} = \infty - \infty$ ⇝ manipulate

Common denom: $\lim_{x\to1^+} \dfrac{\ln(x)-(x-1)}{(x-1)\ln(x)} = \lim_{x\to1^+} \dfrac{\ln(x)-x+1}{(x-1)\ln(x)} = \dfrac{\ln(1)-1+1}{(1-1)\ln(1)} = \dfrac{0}{0}$ ⇝ Good for l'H

$$\hookrightarrow = \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{\ln(x)+(x-1)\cdot\frac{1}{x}} = \lim_{x\to 1^+} \frac{\frac{1-x}{x}}{\ln(x)+\frac{x-1}{x}} = \lim_{x\to 1^+} \frac{\frac{1-x}{x}}{\frac{x\ln x + x-1}{x}}$$

$$= \lim_{x\to 1^+} \frac{1-x}{x\ln x + x - 1} = \frac{1-1}{1\cdot\ln(1)+1-1} = \frac{0}{0}$$

Again l'Hôp

$$= \lim_{x\to 1^+} \frac{-1}{\ln x + \frac{x}{x} + 1} = \frac{-1}{\ln(1)+1+1} = \boxed{\frac{-1}{2}}$$

d) $$\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}} \overset{\text{sub}}{=} \frac{\sqrt{\infty}}{\sqrt{\infty}} = \frac{\infty}{\infty}$$ ⇝ indeterminate

l'Hôp

$$= \lim_{x\to\infty} \frac{\frac{9}{2\sqrt{9x+1}}}{\frac{1}{2\sqrt{x+1}}} = \lim_{x\to\infty} \frac{9\sqrt{x+1}}{\sqrt{9x+1}} = \lim_{x\to\infty} \frac{\frac{9}{2\sqrt{x+1}}}{\frac{9}{2\sqrt{9x+1}}} = \lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$$ ⇝ we got the original functio

Let's try something else:

Factor out the highest degree term:

$$\lim_{x\to\infty} \frac{\sqrt{9x(1+\underbrace{\cancel{\tfrac{1}{9x}}}_{0})}}{\sqrt{x(1+\underbrace{\cancel{\tfrac{1}{x}}}_{0})}} = \lim_{x\to\infty} \frac{\sqrt{9x}}{\sqrt{x}} = \boxed{\sqrt{9} = 3}$$

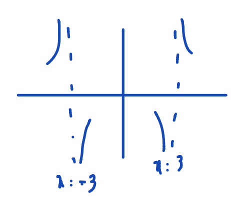
Curve Sketching

b) $f(x) = \dfrac{x^2+1}{x^2-9}$

Domain: $x^2-9=0 \Rightarrow x^2=9 \Rightarrow \underbrace{x=3,-3}_{\text{possible VA}} \Rightarrow$ Domain: $x\neq 3,\ x\neq -3$

VA: $\displaystyle\lim_{x\to 3^+} \frac{x^2+1}{(x-3)(x+3)} = \frac{9+1}{6\cdot 0^+} = \frac{10}{0^+} = \infty$, $\displaystyle\lim_{x\to 3^-} \frac{x^2+1}{(x-3)(x+3)} = -\infty$

Similarly: $\displaystyle\lim_{x\to -3^+} \frac{x^2+1}{(x-3)(x+3)} = \frac{9+1}{-6\cdot 0^+} = -\infty$, $\displaystyle\lim_{x\to -3^-} \frac{x^2+1}{(x-3)(x+3)} = \infty$



So $x=3$, $x=-3$ are VA.

You should write down this verification in the exam.

HA: $\lim_{x\to\infty} \frac{x^2+1}{x^2-9} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hôp}} \boxed{\lim_{x\to\infty} \frac{2x}{2x} = 1}$

Also $\lim_{x\to-\infty} f(x) = 1 \quad \Rightarrow \quad y=1$ is HA

Intercepts

$\to$ y-int $\xrightarrow{x=0}$ $y = \frac{0+1}{0-9} = -\frac{1}{9} \to \left(0, -\frac{1}{9}\right)$

$\to$ x-int $\xrightarrow{y=0}$ $0 = \frac{x^2+1}{x^2-9} \Rightarrow x^2+1=0 \Rightarrow x^2=-1 \Rightarrow$ impossible

Now we go to f':

$$f(x) = \frac{x^2+1}{x^2-9} \Rightarrow f'(x) = \frac{2x(x^2-9)-2x(x^2+1)}{(x^2-9)^2} = \frac{-18x-2x}{(x^2-9)^2} = \frac{-20x}{(x^2-9)^2}$$

$f'(x)=0$

$\Rightarrow -20x=0 \Rightarrow x=0$

f' undefined

$\Rightarrow$ at $x=3$, $x=-3 \Rightarrow$ NOT in the domain

$\hookrightarrow$ Not a critical number

x	-4	-3	-1	0	1	3	4
f'	+	∞	+	0	−	∞	−
f	↗	VA	↗	local max	↘	VA	↘

$f'(x) = \frac{-2x}{\underbrace{(x^2-9)^2}_{\text{always } \oplus}}$

$$f''(x) = \frac{-20(x^2-9)^2 - (-20x)\cdot(2x)(2(x^2-9))}{(x^2-9)^2} = \frac{(x^2-9)\left(-20(x^2-9)+80x^2\right)}{(x^2-9)^4}$$

$\Rightarrow f''(x) = \frac{60x^2+180}{(x^2-9)^3} \Rightarrow f''(x)=0 \Rightarrow 60x^2+180=0 \Rightarrow$ NEVER

$\hookrightarrow f''$ undefined at $x=3$, $x=-3 \to$ NOT in domain

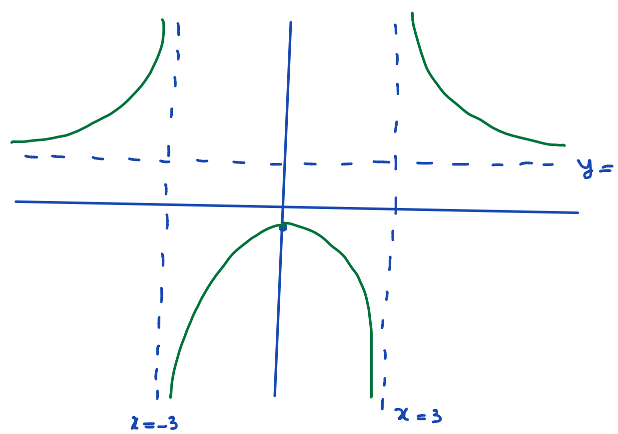
$f''(x) = \frac{60x^2+180}{(x^2-9)^3}$ $60x^2+180 \to$ always $\oplus$

x		-3		3	
f''	+	∞	−	∞	+
f	∪	VA	∩	VA	∪

NO INF Point

Summary Chart

x		-3		0		3	
f'	+ ↗		+ ↗		− ↘		− ↘
f''	+ ∪		− ∩		− ∩		+ ∪
f							



$f(x) = \dfrac{e^x}{1+e^x}$, Domain: $1+e^x = 0 \Rightarrow e^x = -1 \Rightarrow$ NEVER (e^x is always $\oplus$)

$\Rightarrow$ Domain: $\mathbb{R}$

$f(x)$ is defined everywhere, so there is NO VA.

HA: $\lim\limits_{x\to\infty} \dfrac{e^x}{1+e^x} = \dfrac{e^\infty}{1+e^\infty} = \dfrac{\infty}{\infty} \xrightarrow{\text{L'Hôp}} \lim\limits_{x\to\infty} \dfrac{e^x}{e^x} = 1 \Rightarrow y=1$ HA on the right hand side of x-axis

$\lim\limits_{x\to-\infty} \dfrac{e^x}{1+e^x} = \dfrac{e^{-\infty}}{1+e^{-\infty}} = \dfrac{0}{1+0} = 0 \Rightarrow y=0$ HA on the left hand side of x-axis.

intercepts: x-int: $y=0 \Rightarrow 0 = \dfrac{e^x}{1+e^x} \Rightarrow e^x = 0 \Rightarrow$ NEVER

y-int: $x=0 \Rightarrow y = \dfrac{e^0}{1+e^0} = \dfrac{1}{1+1} = \dfrac{1}{2} \Rightarrow (0, \frac{1}{2})$

$f'(x) = \dfrac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2} = \dfrac{e^x}{(1+e^x)^2}$ ⇝ Always positive

$\Downarrow$

f is everywhere increasing.

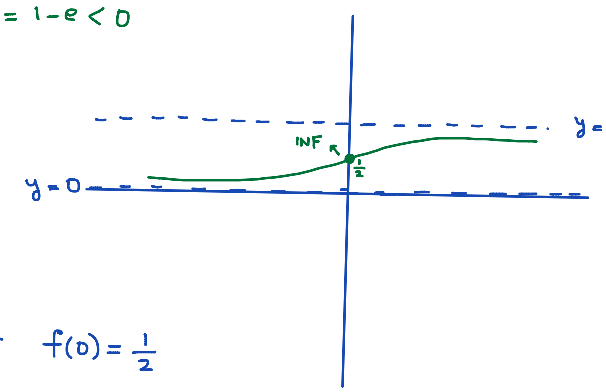
$$f''(x) = \frac{e^x(1+e^x)^2 - e^x \cdot 2(1+e^x)\cdot e^x}{(1+e^x)^4} = \frac{e^x(1+e^x)(1+e^x-2e^x)}{(1+e^x)^4}$$

$$f''(x) = \frac{\overset{+}{e^x}(1-e^x)}{\underset{+}{(1+e^x)^3}}$$

$f''(x) = 0$ when $1-e^x = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$

$1 - \frac{1}{e}$ = test -1 ; test $1 = 1-e < 0$

x		0	
f''	$+$	0	$-$
f	$\cup$	INF point	$\cap$



Summary chart

x		0	
f'	$+\nearrow$	:	$+\nearrow$
f''	$+\cup$	:	$-\cap$
f			

$f(0) = \frac{1}{2}$

Approximation

(3) Let $h(x)$ be the differentiable function such that $h(4.5) = 8.1$ and $h'(4.5) = -7.7$. a) Determine the equation of the tangent line at $x = 4.5$. b) Use part (a) to estimate $h(4.51)$.

a) Equation of the tangent line: $y = h(a) + h'(a)(x-a)$

$a = 4.5$, $h(a) = 8.1$, $h'(a) = -7.7$

$$\Rightarrow \boxed{y = 8.1 - 7.7(x - 4.5)} = L(x)$$

b) $h(4.51) \approx L(4.51) = 8.1 - 7.7(\underbrace{4.51 - 4.5}_{0.01}) = 8.1 - 0.77 = 7.33$

(4f) Estimate $(21)^4$

$\underbrace{f(x) = x^4 \quad \text{nice point}: x = 20}_{\text{Find the tangent line at } x=20}$ $\Rightarrow f'(x) = 4x^3 \Rightarrow f'(20) = 4\cdot 20^3 = 4\cdot 8000 = 32000$

$f(a) = 20^4 = 160000$

$$\Rightarrow L(x) = f(a) + f'(a)(x-a)$$

$$\Rightarrow L(x) = 160000 + 32000(x - 20)$$

Now: bad point: $x = 21$

$$(21)^4 = f(21) \approx L(21) = 160000 + 32000(\underbrace{21-20}_{1}) = 192000$$

(8) In question (6a) above sketch the function and its quadratic approx. Use your approx. to estimate $e^{-0.5}$, by checking the graph determine whether it's an over/under estimate.

a) $f(x) = e^x$ about $x = 0$

$f'(x) = e^x = f''(x) \overset{a=0}{\Rightarrow} f(0) = e^0 = 1$

$f'(0) = 1$, $f''(0) = 1$

$$Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

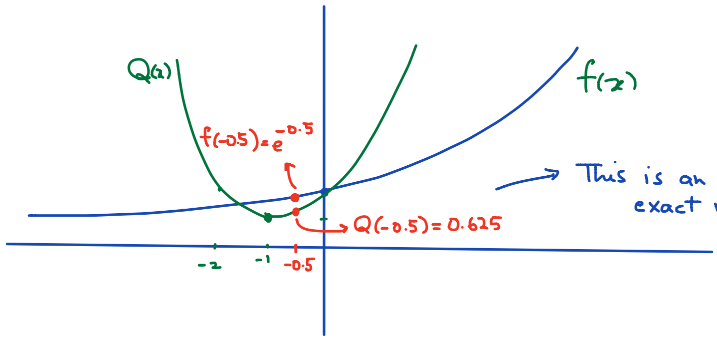
$$\Rightarrow Q(x) = 1 + 1(x-0) + \frac{1}{2}(x-0)^2 \Rightarrow Q(x) = 1 + x + \frac{x^2}{2}$$

Sketch the parabola $y = 1 + x + \frac{x^2}{2}$ and $y = e^x$

x	y
-1	$\frac{1}{2}$
0	1
-2	$1-2+2=1$

Vertex of parabola: $x = \dfrac{-1}{2 \cdot \frac{1}{2}} = -1$ $\Rightarrow$ $(-1, \frac{1}{2})$

$\Rightarrow$ $y = 1 - 1 + \frac{1}{2} = \frac{1}{2}$



This is an underestimate of the exact value.

$$e^{-0.5} = f(-0.5) \approx Q(-0.5) = 1 + (-0.5) + \frac{(\cancel{-0.5})^2}{2} \quad 0.25$$

$$= 1 - 0.5 + 0.125 = 1 - 0.375 = 0.625$$

$$\Rightarrow \quad e^{-0.5} \approx 0.625$$

(9) If the Taylor polynomial of degree 3 of the function $f(x)$ is $T_3(x) = 7x + 3x^3$ near $x = 0$. Find $f'(0)$, $f''(0)$ and $f'''(0)$.

$T_3(x) = 7x + 3x^3$, On the other hand, from Taylor formula we know that

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

So Compare: $f(0) + f'(0) \cdot x + \dfrac{f''(0)}{2} x^2 + \dfrac{f'''(0)}{6} x^3$

with $7x + 3x^3$

equate the coefficients of each term:

$f(0) = 0$

$f'(0) = 7$

$\dfrac{f''(0)}{2} = 0 \Rightarrow f''(0) = 0$

$\dfrac{f'''(0)}{6} = 3 \Rightarrow f'''(0) = 18$

Anti derivative

2) Suppose h is a function such that $h'(x) = x^2 + 2e^x + 3$ and $h(3) = 0$. What is $h(1)$?

$h'(x) = x^2 + 2e^x + 3 \Rightarrow h(x) = \frac{1}{3}x^3 + 2e^x + 3x + C$

$\xrightarrow{h(3)=0} 0 = \frac{1}{3}(3)^3 + 2e^3 + 3.3 + C$

$\Rightarrow 0 = 9 + 2e^3 + 9 + C$

$\Rightarrow C = -18 - 2e^3$

$\Rightarrow h(x) = \frac{1}{3}x^3 + 2e^x + 3x - 18 - 2e^3$

$\Rightarrow \boxed{h(1) = \frac{1}{3} + 2e + 3 - 18 - 2e^3}$

3) Find f if $f''(t) = 2e^t + 3\sin t$ and $f(0) = 0$, $f'(0) = 0$

$f'(t) = 2e^t - 3\cos t + C \xrightarrow{f'(0)=0} 0 = 2e^0 - 3\cos 0 + C \Rightarrow 0 = 2 - 3 + C$

$f(t) = 2e^t - 3\sin t + Ct + D \qquad \Rightarrow \boxed{C = 1}$

$f(t) = 2e^t - 3\sin t + t + D$

$\xrightarrow{f(0)=0} 0 = 2e^0 - 3\sin 0 + 0 + D \Rightarrow 0 = 2 + D \Rightarrow \boxed{D = -2}$

$\Rightarrow \boxed{f(t) = 2e^t - 3\sin t + t - 2}$

5) Find a function $y = f(x)$ with the following properties: and $f(-1) = 0$.

- $y'' = 6x$
- Graph of f passes through the point $(6,1)$ and has a horizontal tangent line at this point.

$f(6) = 1$

$f'(6) = 0$

$$\begin{cases} f''(x) = 6x \\ f(6) = 1 \\ f'(6) = 0 \end{cases} \Longrightarrow \text{The rest is similar to Q (3)}$$

4) (a) Find g if $g''(x) = \frac{1}{\sqrt[3]{x}} - \frac{3}{x^3} - \frac{1}{2}$ and $g(1) = \frac{1}{2}$ and $g(-1) = 0$.

$$g''(x) = x^{-\frac{1}{3}} - 3x^{-3} - \frac{1}{2}$$

$$g'(x) = \frac{1}{\cancel{-\frac{1}{3}+1}_{\frac{2}{3}}} x^{-\frac{1}{3}+1} - 3 \cdot \frac{1}{\cancel{-3+1}_{-2}} x^{-3+1} - \frac{1}{2}x + C$$

$$g'(x) = \frac{3}{2} x^{2/3} + \frac{3}{2} x^{-2} - \frac{1}{2}x + C$$

$$g(x) = \frac{3}{2} \cdot \frac{1}{\cancel{\frac{2}{3}+1}_{\frac{5}{3}}} x^{\frac{2}{3}+1} + \frac{3}{2} \cdot \frac{1}{\cancel{-2+1}_{-1}} x^{-2+1} - \frac{1}{2} \cdot \frac{1}{2}x^2 + Cx + D$$

$$g(x) = \frac{3}{2} \cdot \frac{3}{5} x^{5/3} - \frac{3}{2} x^{-1} - \frac{1}{4}x^2 + Cx + D$$

$g(1) = \frac{1}{2} \longrightarrow \frac{1}{2} = \frac{9}{10} - \frac{3}{2} - \frac{1}{4} + C + D$

$g(-1) = 0 \longrightarrow 0 = -\frac{9}{10} + \frac{3}{2} - \frac{1}{4} - C + D$

$\Rightarrow$ $\frac{1}{2} = \frac{18 - \cancel{30} - 5}{20} + C + D$ $\quad -\frac{17}{20}$

$0 = \frac{-18 + 30 - 5}{20} - C + D$

$$\Rightarrow \begin{cases} C + D = \frac{27}{20} \\ -C + D = -\frac{7}{20} \Rightarrow D = -\frac{7}{20} + C \end{cases}$$

$\longrightarrow C + \left(-\frac{7}{20} + C\right) = \frac{27}{20}$

$$2C - \frac{7}{20} = \frac{27}{20} \Rightarrow 2C = \frac{34}{20} \Rightarrow C = \frac{34}{40}$$

$$\Rightarrow D = -7/20 + \frac{34}{40} = \frac{20}{40} = \frac{1}{2}$$

$$\boxed{g(x) = \frac{3}{2} \cdot \frac{3}{5} x^{5/3} - \frac{3}{2} x^{-1} - \frac{1}{4} x^{2} + \frac{34}{40} x + \frac{1}{2}}$$

↓

Check if my algebra is correct :)