Review Notes (Final Exam)

l'Hôpital's Rule : This is a rule to find indeterminate limits.

The first step is to check whether the limit is indeterminater or not.

↳ Start with direct substitution, if you get:

$\infty - \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$ ⇝ indeterminate forms

Note that:

$$\left.\begin{aligned} \infty \cdot \infty &= \infty \\ 0 \cdot 0 &= 0 \\ \infty + \infty &= \infty \\ \frac{\text{number}}{0} &= \infty \\ \frac{\text{number}}{\infty} &= 0 \end{aligned}\right\} \text{These are NOT indeterminate.}$$

→ How to get rid of indeterminate limits?

↓ manipulate the function do some algebra tricks and again substitute

↓ It's possible you get rid of indeterminate form without any l'Hôpital, especially in $\infty - \infty$ case.

↓ **Apply l'Hôpital**

↓ It can only be applied where there's a fraction in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If you get $0 \cdot \infty$ case, you should first rewrite it as a fraction by applying the trick.

Trick for $\infty \cdot 0$:

$$a \cdot b = \frac{a}{\frac{1}{b}} \overset{\text{or}}{=} \frac{b}{\frac{1}{b}}$$

* Note that in some cases, only one of the options $\frac{a}{\frac{1}{b}}$ or $\frac{b}{\frac{1}{a}}$ works. If you tried one and it doesn't work, switch to the other.

L'Hôpital's Rule :

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\longrightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Replace top and bottom by their derivatives.

This is NOT Quotient Rule.

* Sometimes, you need tha apply the rule more than once, until you get rid of the indeterminate form.

Examples. Evaluate the following limits.

a) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

b) $\lim_{x \to 0} \frac{\sin x}{1+2x}$

c) $\lim_{x \to \infty} \frac{e^{2x} + x^2}{e^x + 4x}$

d) $\lim_{x \to \infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$

e) $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$

f) $\lim_{x \to -\infty} x e^x$

g) $\lim_{x \to \infty} x e^x$

h) $\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^2}$

i) $\lim_{x \to 0^+} \ln(x) \cdot \tan(x)$

j) $\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x)$

k) $\lim_{x \to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$

Curve Sketching

↳ long questions with several steps to follow.
↳ In the exam, the question is probably split into separate parts but if you're given a single question to hand-sketch you should follow the following steps.

- Check the following from the function itself: $f(x)$.
  - → Domain
  - → x and y intercepts ⇝ depending on the given function, they might not be computable.

    x-int ⇝ $y=0$ ⇒ In some cases, NOT do-able without a calculator.

    y-int ⇝ $x=0$ if 0 is in the domain of the function.

    For example; $y = x^3 + x^2 + 2$

    $\xrightarrow{x\text{-int}}$ $0 = x^3 + x^2 + 2$ ⇝ We can NOT solve this.

    $\xrightarrow{y\text{-int}}$ $x = 0 \Rightarrow y = 0 + 0 + 2 = 2 \Rightarrow (0,2)$

  - → Vertical & Horizontal Asymptotes

    ↓ These are coming from the original function and not its derivative

    - Know the definition and explicitly verify it in the exam.

* $x=a$ is VA of f if $\lim_{x \to a} f(x) = +\infty$ or $-\infty$

↓ Possible candidates are the ones that are excluded from the domain (such as roots of the denominator ...)

For every possible asymptote you should clearly verify the definition.

* $y = b$ is HA of f if $\lim_{\substack{x \to \infty \\ \text{or } x \to -\infty}} f(x) = b$

To verify the limits for HA (and VA) in some cases you should apply l'Hôpital's rule, so you may need to take the derivative for l'H although asymptotes themselves are NOT about derivatives.

- Check the following for the derivative : $f'(x)$

→ Find $f'(x)$ and use it to find critical numbers of f

$$\underbrace{\text{NOT defined} \quad \overset{f'(x)}{\longleftrightarrow} \quad = 0}_{\text{critical numbers as long as they are in the domain.}}$$

→ Make a sign chart for f'

x	here sits the critical numbers and the points that are excluded from the domain. (in an increasing order)
f'	test points between each number in the first row. Plug the test points into f' and determine if it's ⊕ or ⊖
f	

Some terms are always ⊕ or ⊖, don't check these.

1st derivative test

If f' is ⊕ then f ↗

If f' is ⊖ then f ↘

↗ • ↘ → local max

↘ • ↗ → local min

→ If you are asked to write intervals of increase / decrease, make sure you use (or) for any point that's not in the domain and for $\pm\infty$.

- Check the following for the 2nd derivative: $f''(x)$

→ $f''(x)$ is used for concavity.

$f''(x) = 0$ $f''(x)$ NOT defined (but in the domain)

These are NOT INF points. There must be a sign change in f'' as well.

→ A sign chart for f''

x	
f''	
f	

Here sits the points we found above, including the points that are NOT in the domain.

test points, plug them into f'' and find the sign for f''.

If f'' is ⊕ f is $\cup$

If f'' is ⊖ f is $\cap$

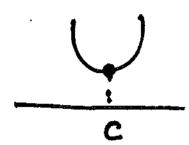
If f goes from $\cup$ to $\cap$ or $\cap$ to $\cup$ ⇒ there is an inflection point.

- Make a summary chart of everything combined f', f'', f. Determine the shape of the function at each interval.

- Sketch the graph. Be careful about asymptotes and how the function should reach to its asymptotes ⇝ Don't forget y-coordinate

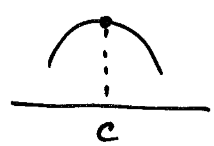
- Don't forget to be efficient in your computations. Especially, when finding $f''(x)$, make sure you choose the simplest form of f' to derive. Try to write fractions as negative powers as much as possible. When we're solving equations, it's easier to first factor the common terms out, but when differentiating it's easier to derive the un-factored version.

- If the question just wants you to find local min/max and not INF point or concavity, check if it's easier to apply the 2^{nd} derivative test $\rightsquigarrow$ take $f''(x)$ and evaluate it at each of the critical numbers say $x=c$ if $f''(c)>0$



$\Rightarrow$ f has a local min at $x=c$.

if $f''(c)<0$



$\Rightarrow$ f has a local max at $x=c$.

Examples. Sketch the following functions.

a) $f(x) = x^3 - 6x^2 + 9x + 1$

b) $f(x) = \dfrac{x^2+1}{x^2-9}$

c) $g(t) = \dfrac{t^2+5t+1}{t^2}$

d) $f(x) = \dfrac{e^x}{1+e^x}$

f) $h(x) = \ln(4-x^2)$

g) $f(r) = r(r-2)^{2/3}$

h) $f(x) = \dfrac{x}{x^2-9}$

i) $f(x) = x\,e^{-x}$

Optimization

. The key strategy is to read the question carefully, find the quantity that should be maximized or minimized. You write down the formula for that function perimeter, area, volume, cost ... . There are often more than one variable in your formula, so you should check the question to find extra information so that to reduce the function to only a function of single variable. You then differentiate and check whether the critical numbers are max or min.

Things to Remember :

- Draw a diagram/shape for the given situation and label your diagram with letters.
- Note that sometimes the fixed values are themselves given with letters (parameters). Don't confuse the variables with fixed values especially when differentiating.
- When you're done differentiating, don't forget to check
  (1) the critical number is whether local min/max. Do this by apply 1^{st} derivative test or 2^{nd} derivative test, whichever is easier

(2) Finally check if the local max/min is global min/max. To this you need to find the domain in the context of the question :

→ Closed interval : Check the y-value for each endpoint, compare with the y-value of the critical number.

→ Open interval : Visualize the graph of the function, if it's always concave up or concave down , then the critical number is the global extrema.

→ WebWork questions are great for practice.

2. Find the absolute maximum and minimum values of $f(x) = x^3 - 12x - 5$ on the interval $[-4, 6]$. Clearly explain your reasoning.

3. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$.

~~4.~~ ~~Find all critical points of the functio~~n $f(x) = |3x - 5|$ on the interval $[-3, 2]$. ~~Also find all maxima and minima of this function o~~n $[-3, 2]$, ~~both local and global.~~

5. The sum of two positive numbers is 12. What is the smallest possible value of the sum of their squares? Show your reasoning.

6. If a and b are positive numbers, find the x coordinate which gives the absolute maximum value of $f(x) = x^a(1-x)^b$ on the interval $[0, 1]$.

7. Find the point on the curve $x + y^2 = 0$ that is closest to the point $(0, -3)$.

8. A straight piece of wire 40 cm long is cut into two pieces. One piece is bent into a circle and the other is bent into a square. How should wire be cut so that the total area of both circle and square is **minimized**?

9. A straight piece of wire of 28 cm is cut into two pieces. One piece is bent into a square (i.e., dimensions x times x.) The other piece is bent into a rectangle with aspect ratio three (i.e., dimensions y times $3y$.)

   What are dimensions, in centimeters, of the square and the rectangle such that the sum of their areas is **minimized**?

10. With a straight piece of wire 4m long, you are to create an equilateral triangle and a square, or either one only. Suppose a piece of length x meters is bent into triangle and the reminder is bent into a square. Find the value of x which maximizes the total area of both triangle and square.

11. A rectangle with sides parallel to the coordinate axes is to be inscribed in the region enclosed by the graphs of $y = x^2$ and $y = 4$ so that its perimeter has maximum length.

    (a) Sketch the region under consideration.

    (b) Supposing that the x-coordinate of the bottom right vertex of the rectangle is a, find a formula which expresses P, the length of the perimeter, in terms of a.

    (c) Find the value of a which gives the maximum value of P, and explain why you know that this value of a gives a maximum.

(d) What is the maximum value of P, the length of the perimeter of the rectangle?

12. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 12 - x^2$.

13. A farmer has 400 feet of fencing with which to build a rectangular pen. He will use part of an existing straight wall 100 feet long as part of one side of the perimeter of the pen. What is the maximum area that can be enclosed?

14. A $10\sqrt{2}$ ft wall stands 5 ft from a building, see Figure 3.4. Find the length L of the shortest ladder, supported by the wall, that reaches from the ground to the building.

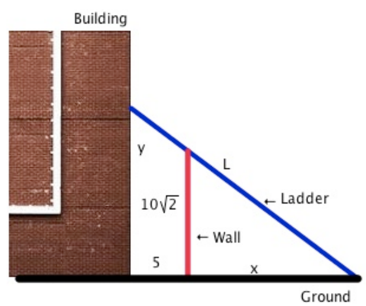


Figure 3.4: Building, Wall, and Ladder

15. In an elliptical sport field we want to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the length $2x$ and width $2y$ of the pitch (in terms of a and b) that maximize the area of the pitch. [Hint: express the area of the pitch as a function of x only.]

16. The top and bottom margins of a poster are each 6 cm, and the side margins are each 4 cm. If the area of the printed material on the poster (that is, the area between the margins) is fixed at 384 cm^2, find the dimensions of the poster with the smallest total area.

17. Each rectangular page of a book must contain 30 cm^2 of printed text, and each page must have 2 cm margins at top and bottom, and 1 cm margin at each side. What is the minimum possible area of such a page?

18. Maya is 2 km offshore in a boat and wishes to reach a coastal village which is 6 km down a straight shoreline from the point on the shore nearest to the boat. She can row at 2 km/hr and run at 5 km/hr. Where should she land her boat to reach the village in the least amount of time?

19. A rectangular box has a square base with edge length x of at least 1 unit. The total surface area of its six sides is 150 square units.

    (a) Express the volume V of this box as a function of x.

    (b) Find the domain of $V(x)$.

    (c) Find the dimensions of the box in part (a) with the greatest possible volume. What is this greatest possible volume?

20. An open-top box is to have a square base and a volume of 10 m^3. The cost per square meter of material is \$5 for the bottom and \$2 for the four sides. Let x and y be lengths of the box's width and height respectively. Let C be the total cost of material required to make the box.

    (a) Express C as a function of x and find its domain.

    (b) Find the dimensions of the box so that the cost of materials is minimized. What is this minimum cost?

21. An open-top box is to have a square base and a volume of 13500 cm^3. Find the dimensions of the box that minimize the amount of material used.

22. ~~A water trough is to be made from a long strip of tin 6 ft wide by bending up at an the angle θ a 2 ft strip at each side. What angle θ would maximize the cross sectional area, and thus the volume, of the trough? (See Figure 3.5.)~~

23. Find the dimension of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius R and height H.

24. A hollow plastic cylinder with a circular base and open top is to be made and 10 m^2 plastic is available. Find the dimensions of the cylinder that give the maximum volume and find the value of the maximum volume.

25. An open-topped cylindrical pot is to have volume 250 cm^3.The material for the bottom of the pot costs 4 cents per cm^2; that for its curved side costs 2 cents per cm^2. What dimensions will minimize the total cost of this pot?

26. Cylindrical soup cans are to be manufactured to contain a given volume V. No waste is involved in cutting the material for the vertical side of each can, but each top and bottom which are circles of radius r, are cut from a square that measures $2r$ units on each side. Thus the material used to manufacture each soup can has an area of $A = 2\pi rh + 8r^2$ square units.

    (a) How much material is wasted in making each soup can?

    (b) Find the ratio of the height to diameter for the most economical can (i.e. requiring the least amount of material for manufacture.)

    (c) Use either the first or second derivative test to verify that you have minimized the amount of material used for making each can.

27. A storage container is to be made in the form of a right circular cylinder and have a volume of 28π m^3. Material for the top of the container costs \$5 per square metre and material for the side and base costs \$2 per square metre. What dimensions will minimize the total cost of the container?

28. Show that the volume of the largest cone that can be inscribed inside a sphere of radius R is $\dfrac{32\pi R^3}{81}$.

~~29. The sound level measured in watts per square meter, varies in direct proportion to the power of the source and inversely as the square of the distance from the~~

Approximation

Linear Approximation $\overset{\text{at "a"}}{\rightsquigarrow}$ tangent line : $L(x) = f(a) + f'(a).(x-a)$

Quadratic Approx. $\rightsquigarrow$ parabola : $Q(x) = f(a) + f'(a).(x-a) + \frac{f''(a)}{2}.(x-a)^2$

Higher degree Approx $\rightsquigarrow$ Taylor polynomial : $T_n(x)$

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3 things to distinguish

- A function : $f(x)$
  - $\rightarrow$ either explicitly given
  - $\rightarrow$ or you need to come up with a function based on the given value.

- A "nice" point at which we find the tangent line or higher degree polynomials. This is the number "a" in the above formulas $\rightsquigarrow$ This number is usually without any decimals.

- A "bad" point which is used at the last step of our solution $\rightsquigarrow$ Usually comes with decimal points.

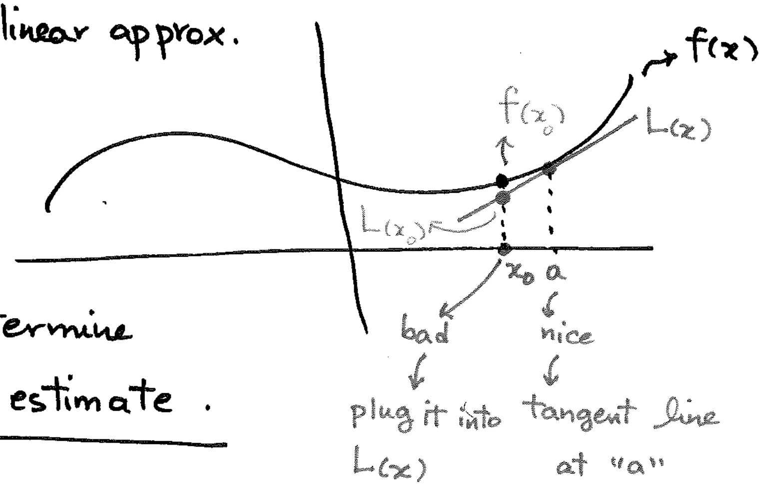
  We find our approximation by using the "nice" number and at the last step, we plug in the "bad" number into our approximation and find an estimate.

$\longrightarrow$ You may be asked to sketch the function and its approx.

→ Make sure you understand which one is the "nice" point and which one is the "bad" point and how each one should be used.

For example, in the case of linear approx.

$$f(x_0) \approx L(x_0)$$



→ You may be asked to determine the underestimate or over-estimate.

The estimated value is smaller than the exact value.

For example; $f(x_0) > L(x_0)$

The estimated value is larger than the exact value.

$f(x_0) < L(x_0)$

You can determine over/under-estimate by checking the graph as above, or they might give you the exact value and ask you to find the approx. value and then decide.

→ The higher degree polynomials are made by adding more terms to the linear approximation, and we denote it by $T_n(x)$, where n is the degree of the polynomial.

$$T_1(x) = L(x) = f(a) + f'(a)\cdot(x-a)$$

$$T_2(x) = Q(x)\quad f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$T_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

⋮ Keep the pattern

$$T_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots + \frac{f^{(10)}(a)}{10!}(x-a)^{10} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$2! = 2\cdot 1 = 2$

$3! = 3.2.1 = 6$

notation for 4th derivative of f evaluated at a (nice point)

10^{th} derivative at "a"

$10! = 10.9.8.7.\ \ldots .1 = ?$ not important to evaluate

nth derivative of f at a

$4! = 4.3.2.1 = 24$

$n! = n.(n-1).(n-2).\ \ldots .\ 3.2.1$

Practice Problems.

(1) Find the linear approx. to the following function near the given point.

a) $f(x) = x^3 - x$ near $x = 3$

b) $f(x) = 3x e^{2x-10}$ near $x = 5$

c) $g(z) = \sqrt[4]{z}$ near $z = 1$

d) $f(t) = \cos(2t)$ at $t = \frac{1}{2}$

e) $f(x) = \sqrt{1-x}$ at $x = 0$

(2) a) Use (1c) to approximate the value of $\sqrt[4]{2}$ and $\sqrt[4]{15}$.

b) Without using a calculator, determine if your approximation works better for $\sqrt[4]{2}$ or for $\sqrt[4]{15}$. Justify your answer.

c) How can you modify your approximation to make it more accurate for your answer given in part (b).

(3) Let $h(x)$ be the differentiable function such that $h(4.5) = 8.1$ and $h'(4.5) = -7.7$. a) Determine the equation of the tangent line at $x = 4.5$. b) Use part (a) to estimate $h(4.51)$.

(4) Estimate the following values :

(a) $\sqrt{80}$ (c) $e^{0.1}$ (e) $\ln(3)$

(b) $(63)^{2/3}$ (d) $\ln(1.2)$ (f) $(21)^4$

(5) (a) Find the linear approx. to $f(x) = \sqrt[3]{x+8}$ at $x = 0$.

(b) Use this approximation to estimate $\sqrt[3]{7.95}$ and $\sqrt[3]{8.1}$.

(c) If calculator gives $\sqrt[3]{7.98} = 1.995824$ and $\sqrt[3]{8.1} = 2.008298$ Determine for each of your approx. in part (b) whether they're over/under estimate.

(6) Find the quadratic approx. for the following functions.

a) $f(x) = e^x$ about $x = 0$

b) $f(x) = x^4 e^{-3x^2}$ near $x = 0$

c) $f(x) = \cos(x)$ near $x = 0$

d) $f(x) = \ln(x)$ near $x = 2$

(7) (a) Find the 3rd degree Taylor polynomial of $f(x) = \sqrt{x}$ near $x = 0$

(b) Find $T_5(x)$ for $f(x) = \ln(x+1)$ near $x = 0$.

(8) In question (6a) above sketch the function and its quadratic approx. Use your approx. to estimate $e^{-0.5}$, by checking the graph determine whether it's an over/under estimate.

(9) If the Taylor polynomial of degree 3 of the function $f(x)$ is $T_3(x) = 7x + 3x^3$ near $x = 0$. Find $f'(0)$, $f''(0)$ and $f'''(0)$.

Anti derivative. For this topic, I just give the practice problems and since the concept is straightforward just refer to the lecture notes.

Practice Problems.

1) a) Find f if $f'(x) = 2\cos x + 8x^3 - e^3$ and $f(0) = 7$.

b) Find the general form of g if $g'(x) = \sin x + \frac{1}{x^2} - e^x$.

2) Suppose h is a function such that $h'(x) = x^2 + 2e^x + 3$ and $h(3) = 0$. What is $h(1)$?

3) Find f if $f''(t) = 2e^t + 3\sin t$ and $f(0) = 0$, $f'(0) = 0$

4) (a) Find g if $g''(t) = \frac{1}{\sqrt[3]{x}} - \frac{3}{x^3} - \frac{1}{2}$ and $f(1) = \frac{1}{2}$ and $f(-1) = 0$.

5) Find a function $y = f(x)$ with the following properties:

- $y'' = 6x$
- Graph of f passes through the point $(6,1)$ and has a horizontal tangent line at this point.