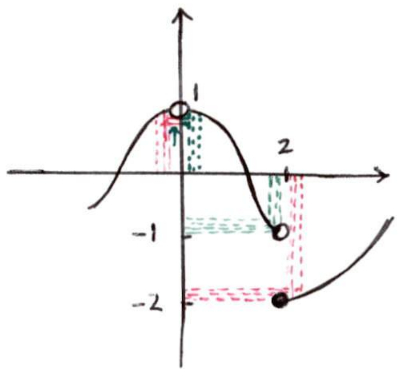
Lecture 10
Sept 26

HW 2 posted. Due date: Monday Oct 1
Collect HW 1 from MLC (LSK 301/302, 11am-5pm)

Clicker Q : Find the following limits using the graph of $f(x)$.



(1) $\lim_{x \to 0} f(x) = 1$ and $f(0)$ = undefined

A. 0 $\boxed{\text{C. 1}}$ E. NOT sure :(
B. 2   D. −1

(2) $\lim_{x \to 2^+} f(x) = -2$

right limit at 2

approaching 2 from right

A. −1   C. 1   E. Again NOT sure :(
$\boxed{\text{B. } -2}$ D. 2

(3) $\lim_{x \to 2^-} f(x) = -1$

left limit at 2

approaching 2 from left

$\boxed{\text{A. } -1}$ C. 1   E. Again !!!
B. −2   D. 2

What is the full limit at 2 then:

$\lim_{x \to 2} f(x)$ = Does NOT exist, because the one-sided limits are NOT equal and we don't get a unique value.

Limit Exists or Does Not Exist (DNE)

Exists : If one-sided limits both from left and right exist and they are equal then the full limit exists, and the value of the limit is just the value of one-sided limits $\rightsquigarrow$ Clicker Q : 1

$$\lim_{x\to 0^+} f(x) = 1 = \lim_{x\to 0^-} f(x) \quad \text{and} \quad \lim_{x\to 0} f(x) = 1$$

The full limit exists and it is equal to the left & right lim

Does Not Exist

DNE : If one-sided limits are NOT equal to each other then the full limit DNE $\rightsquigarrow$ Clicker Q 2 & 3

$$\left.\begin{aligned} \lim_{x\to 2^+} f(x) &= -2 \\ \lim_{x\to 2^-} f(x) &= -1 \end{aligned}\right\} \Longrightarrow \lim_{x\to 2} f(x) = \text{Does Not Exist} = \text{DNE}$$

In math language

If $\displaystyle\lim_{x\to a} f(x) = L$ $\xrightarrow{\text{implies}}$ $\displaystyle\lim_{x\to a^+} f(x) = L$

$\xrightarrow{\text{implies}}$ $\displaystyle\lim_{x\to a^-} f(x) = L$

and vice versa

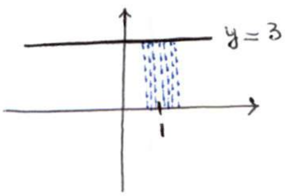
If $\displaystyle\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$ then $\displaystyle\lim_{x\to a} f(x) = L$

$\Rightarrow$ This is a two-way conclusion :

$$\boxed{\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L}$$

So far we could find the limits using the graph of a given function, but what if there's no graph given and we only have the expression for the function. Let's start with two easy examples.

Clicker Q: What is $\lim_{x \to 1} (3) = 3$ $\quad f(x) = 3$

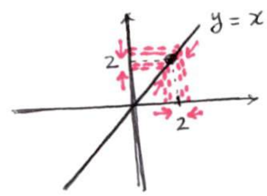


A. 0 $\quad$ B. 1 $\quad$ C. 2 $\quad$ D. 3 $\quad$ E. DNE

* In general if $f(x) = C$ where C is a constant number then

$$\boxed{\lim_{x \to a} C = C} \quad \star$$

Clicker Q: What is $\lim_{x \to 2} x = 2$ $\quad f(x) = x$



$$\boxed{\lim_{x \to a} x = a} \quad \star\star$$

Using the above simple limit and the following "limit laws" we can compute limits of "nice" functions.

Limit Laws: Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

(I) $$\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

(II) $$\lim_{x \to a} \left(f(x) - g(x) \right) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

(III) $$\lim_{x \to a} \left(C f(x) \right) = C \lim_{x \to a} f(x) = C \cdot L$$

C: a constant number

(IV) $$\lim_{x \to a} \left(f(x) \cdot g(x) \right) = \left(\lim_{x \to a} f(x) \right) \cdot \left(\lim_{x \to a} g(x) \right) = L \cdot M$$

(V) $$\lim_{x\to a}\left(\left(f(x)\right)^n\right) = \left(\lim_{x\to a} f(x)\right)^n = L^n$$

(VI) $$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$$

Example 1. Find $\displaystyle\lim_{x\to -2} \frac{x^2-3x+2}{x^3+6}$

* We use the limit laws until we could break everything into $\lim_{x\to a} c$ and $\lim_{x\to a} x$

$$\lim_{x\to -2}\frac{x^2-3x+2}{x^3+6} \overset{\text{VI}}{=} \frac{\lim_{x\to -2} x^2-3x+2}{\lim_{x\to -2} x^3+6}$$

$$\overset{\text{I and II}}{=} \frac{\lim_{x\to -2} x^2 - \lim_{x\to -2} 3x + \lim_{x\to -2} 2}{\lim_{x\to -2} x^3 + \lim_{x\to -2} 6}$$

$$\overset{\text{V}}{=} \frac{\left(\lim_{x\to -2} x\right)^2 - 3\left(\lim_{x\to -2} x\right) + \lim_{x\to -2} 2}{\left(\lim_{x\to -2} x\right)^3 + \lim_{x\to -2} 6} \overset{\star \text{ and } \star\star}{=} \frac{(-2)^2-3(-2)+2}{(-2)^3+6} = -6$$

so the limit value is -6.

Now let's substitute -2 into the function: $\dfrac{(-2)^2-3(-2)+2}{(-2)^3+6} = -6$

So in this example, we can simply substitute the given x value: $$\lim_{x\to -2}\frac{x^2-3x+2}{x^3+6} = \frac{(-2)^2-3(-2)+2}{(-2)^3+6} = -6$$

Example 2: Find $\displaystyle \lim_{x\to 3} \frac{x^2+x-20}{x-4} =$

Let's start with the boring way:

$$\lim_{x\to 3} \frac{x^2+x-20}{x-4} = \frac{\lim_{x\to 3} x^2+x-20}{\lim_{x\to 3} x-4} = \frac{\lim_{x\to 3} x^2 + \lim_{x\to 3} x - \lim_{x\to 3} 20}{\lim_{x\to 3} x - \lim_{x\to 3} 4}$$

$$= \frac{\left(\lim_{x\to 3} x\right)^2 + \lim_{x\to 3} x - \lim_{x\to 3} 20}{\lim_{x\to 3} x - \lim_{x\to 3} 4} = \frac{(3)^2+3-20}{3-4} = \frac{-8}{-1} = 8$$

Now let's try direct substitution:

$$\lim_{x\to 3} \frac{x^2+x-20}{x-4} = \frac{(3)^2+3-20}{3-4} = 8 \checkmark$$

$\boxed{\text{Note}}$:

The 1^{st} step in finding a limit $\rightsquigarrow$ Direct Substitution

If the function is "nice", direct substitution always works.

"Nice" functions include:

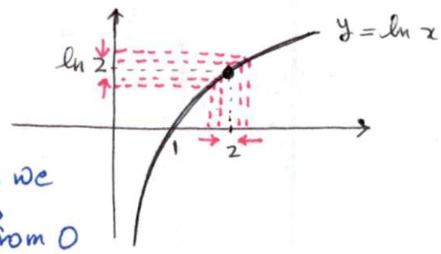
- polynomials (constant)
- rational functions of polynomials with no zero in the denominator.
- $\sin x$, $\cos x$, e^x
- $\sqrt{x}$ and $\log_b x$ (when $x > 0$)

$x \geq 0$ $\quad$ $x > 0$

Example 3: Find the following limits

(1) $\lim_{x \to 2} (\ln x) = \ln 2$

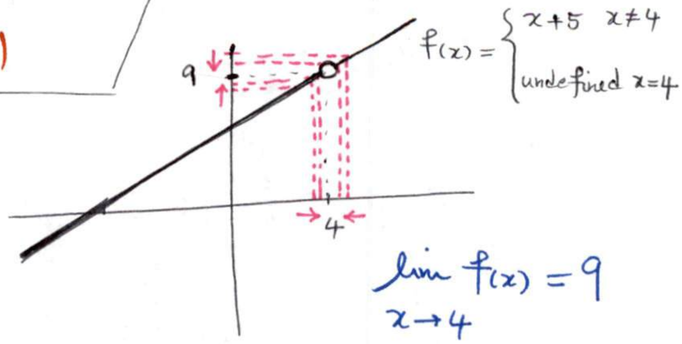
Recall:



*As long as, we are away from 0 everything is nice for $y = \ln x$.

(2) $\lim_{x \to \pi} (\sqrt{x} + \cos x) = \sqrt{\pi} + \cos \pi$

$= \sqrt{\pi} - 1$



$\lim_{x \to 4} f(x) = 9$

What if $f(x)$ is NOT "nice"?

Let's re-do example 2 with limit in another x:

$$\lim_{x \to 4} \frac{x^2 + x - 20}{x - 4} = \frac{(4)^2 + 4 - 20}{4 - 4} = \frac{0}{0} \;!!!$$

always start with substitution

$\frac{0}{0}$ is NOT a number, we call $\frac{0}{0}$ an <u>indeterminate form</u>.

How to remove $\frac{0}{0}$? Manipulate the function doing some algebra

$$\lim_{x \to 4} \frac{x^2 + x - 20}{x - 4} = \lim_{x \to 4} \frac{(x+5)(x-4)}{(x-4)} = \lim_{x \to 4} x + 5 = 4 + 5 = 9$$

x approaches 4, $x \neq 4$

problem found: cancel it

re-substitute.

<u>Question</u>: Write $f(x) = \frac{x^2 + x - 20}{x - 4}$ as a piece-wise function and plot that.

Recall from <u>Lab 1</u>: $f(x) = \frac{(x+5)(x-4)}{x-4} = \begin{cases} x + 5 & x \neq 4 \\ \text{undefined} & x = 4 \end{cases}$ → We don't need to worry about this in limit.

* Since when $x \to 4$ then $x \neq 4$ so we can cancel $x - 4$ without worrying about undefined case.

$\boxed{\text{Note}}$ Re-visit limit evaluation :

1st step : Always start with direct substitution.

If you get a number

Voila ! You have the limit ☺

If you get an indeterminate form NOT a number : $\frac{0}{0}$!!!

manipulate the function by doing algebra (factoring, common denominator, ...) and remove the term which makes top and bottom 0

substitute again ☺

Example 4.

Find $\displaystyle\lim_{x \to -1} \frac{x^2 - 1}{x^2 + x}$

Start with direct substitution:

$$\lim_{x \to -1} \frac{x^2 - 1}{x^2 + x} = \frac{(-1)^2 - 1}{(-1)^2 + (-1)} = \frac{1 - 1}{1 - 1} = \frac{0}{0} \text{ !!!}$$

Factor

$$= \lim_{x \to -1} \frac{(x-1)\cancel{(x+1)}}{x\cancel{(x+1)}}$$

$$= \lim_{x \to -1} \frac{x - 1}{x} = \frac{-1 - 1}{-1} = 2$$