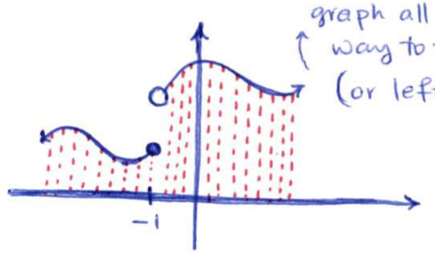
Lecture 3, Sept 10

In this lecture, we continue with functions.

Go back to the example from last lecture where we found 4 graphs to be the graph of functions. Now, let's find the domain of each function.

(*) Recall that domain of a function is the set of all acceptable inputs, namely the x-values that we are allowed to plug into function expression.

(II)



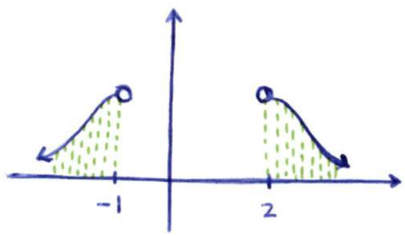
arrows mean graph all the way to right (or left).

To find the domain: trace the x-values and check if there is any x-value for which NO y-value (output) is given. Those x-values are NOT in the domain.

* All x-values are OK, $x=-1$ is also OK because of the filled hole ● which means there is a y-value.

$\Longrightarrow$ Domain: All real numbers $= \mathbb{R} = (-\infty, \infty)$

(III)



* NO x-value between -1 and 2 including -1 and 2 is corresponded to a y-value, so Domain will be:

Domain: $\{ x \in \mathbb{R} : x > 2$ or $x < -1 \}$ ⇝ This is set notation

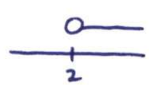
$\in$: Notation for «belongs to»

$= (2, \infty) \cup (-\infty, -1)$

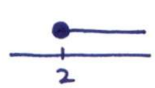
$\cup$ called «union», it joins interval.

Interval Notation:

$(2, \infty)$ ⇝ all real numbers greater than 2 $\xrightarrow[\text{real line}]{\text{on the}}$

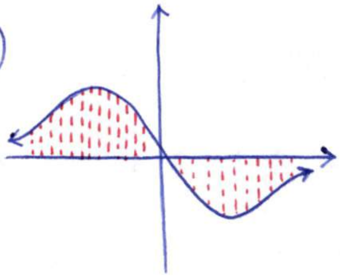


$[2, \infty)$ ⇝ all real numbers greater than 2 including 2 ⇝



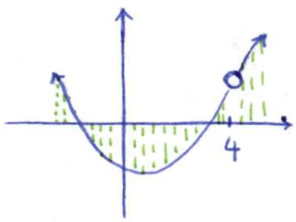
* NEVER close brackets for $\pm\infty$: $[-\infty, 1)$ ✗ , $(2, \infty]$ ✗ , $[-\infty, \infty)$ ✗ , $[3, \infty]$ ✗

(V)



Domain : $\mathbb{R} = (-\infty, \infty)$

(VI)



* For all x-values there is a y-value except for $x = 4$ so

Domain : $\mathbb{R} \setminus \{4\} = \mathbb{R} - \{4\}$

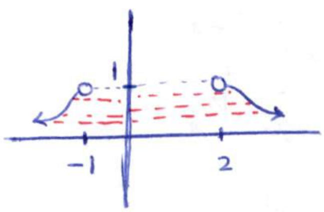
except   except

$= \{ x \in \mathbb{R} : x \neq 4 \}$

$= (-\infty, 4) \cup (4, \infty)$

Now let's pick III and V and find their range.

(III)



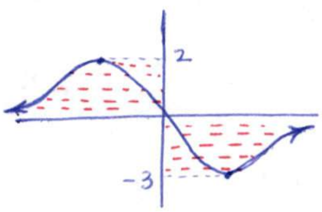
* Assume the function goes down at two ends.

* To find range : trace the y-values and check whether there is any y-value which is NOT an output for some x-value. These y-values are NOT in the range.

Range : All y-values below $y = 1$ and NOT including $y = 1$

$= \{ y \in \mathbb{R} : y < 1 \} = (-\infty, 1)$

(V)



Range : All y-values between -3 and 2, both included

$= \{ y \in \mathbb{R} ; -3 \leq y \leq 2 \} = [-3, 2]$.

Remark : In the above examples, functions II and III are examples of piecewise functions. In fact, the domain can be split into several pieces and in each piece different expressions for the functions can be considered.

Practice Problem: Plot the piecewise function

$$f(x) = \begin{cases} x^2 - 5, & x < 1 \\ 2x - 1, & x > 1 \end{cases}$$

$x = 1$ is NOT included in any of the graphs.

- What is the domain of f?

f has two pieces: 1: $y = x^2 - 5$, we sketch its graph and we only pick the piece for which $x < 1$:

$y = x^2 - 5 \rightsquigarrow$ This is a parabola

vertex: $x = \frac{-b}{2a} = \frac{0}{2} = 0$

$y = 0^2 - 5 = -5 \Rightarrow (0, -5)$

Recall: $y = ax^2 + bx + c$

vertex $\Rightarrow x = -\frac{b}{2a}$

$y = f\left(-\frac{b}{2a}\right)$

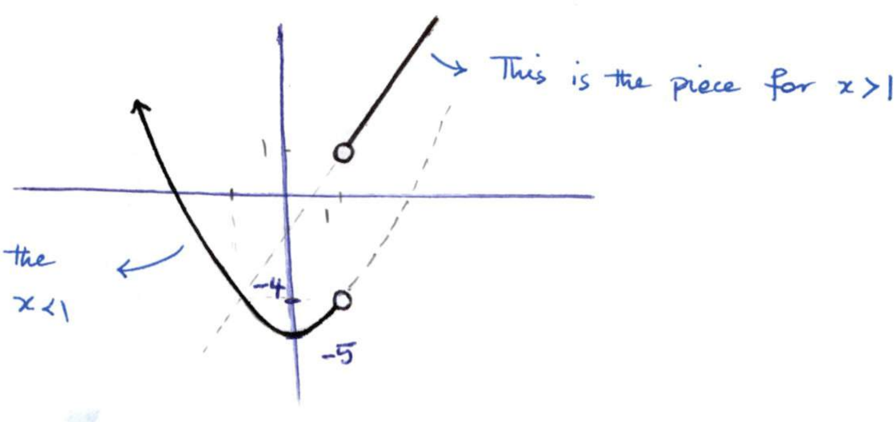
if $a > 0$  $a < 0$ 

* Note that $y = x^2 - 5$ is the graph of $y = x^2$ moved 5 units downward.

2: $y = 2x - 1 \rightsquigarrow$ This is line. We plot the line and take the piece for which $x > 1$.

* To sketch a line we just find two points on the line:

x	0	1
y	$2(0) - 1 = -1$	$2(1) - 1 = 1$



Different types of functions and their domain

1. polynomials $\rightsquigarrow$ The nicest family

Examples : $f(x) = 2x+1$, $y = 3x^2 + 2x$, $y = x^3 - 4x + 5$, $y = 5x^7 + 8x^5 + 2$

Domain : $\mathbb{R}$ $\rightsquigarrow$ All real numbers are OK to be plugged in for x.

$\rightsquigarrow$ polynomials are defined everywhere.

Example : cubic function

$$f(x) = x^3$$

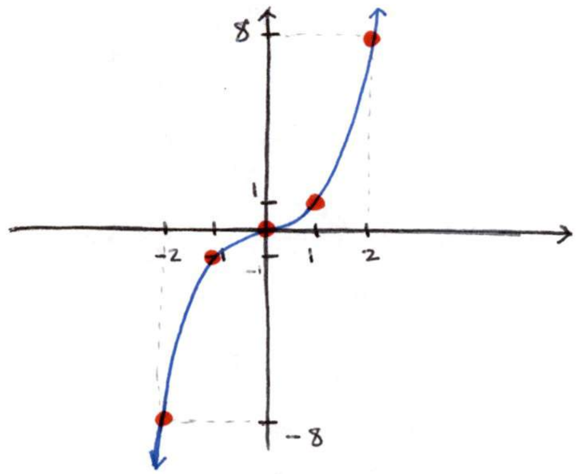
* All x-values are OK, let's find the function value for some of these x's and sketch the function graph.

$f(0) = 0^3 = 0$

$f(1) = 1^3 = 1$, $f(-1) = (-1)^3 = -1$

$f(2) = 2^3 = 8$, $f(-2) = (-2)^3 = -8$

$\vdots$



* Domain $= \mathbb{R}$

* Range $= \mathbb{R}$

2. Rational functions : $\dfrac{\text{Polynomial}}{\text{polynomial}}$

Examples : $y = \dfrac{2x+1}{x-2}$, $y = \dfrac{x^2+2x}{x^3-1}$, $y = \dfrac{x^7+5x^4+5}{2x^3-x^2-4}$

Domain : $\mathbb{R}$ except those x-values that make the denominator $= 0$

Example . $y = \dfrac{2x+1}{x-2}$ Domain : $\mathbb{R}$ except $x=2$ $= \mathbb{R} - \{2\}$

$= (-\infty, 2) \cup (2, \infty)$

* when $x = 2 \rightsquigarrow y = \dfrac{2 \cdot 2 + 1}{2-2} = \dfrac{5}{0} \rightsquigarrow$ undefined !!!

$$y = \frac{x^5 - 2x + 1}{x^2 - 9}$$

Let's find the x-values where $x^2 - 9 = 0$

$x^2 - 9 = (x-3)(x+3) = 0$ $\longrightarrow$ $x - 3 = 0 \Rightarrow x = 3$, $\longrightarrow$ $x + 3 = 0 \Rightarrow x = -3$ } exclude these

Domain : every real number except $x = 3$ or $x = -3$

$= \mathbb{R} - \{-3, 3\}$ $\overset{?}{\Rightarrow}$ Write this in interval notation.

3. Square root functions : $\sqrt{f(x)}$ $\leadsto$ under root must be <u>non-negative</u>

$\Rightarrow$ Domain : $\{x \in \mathbb{R} : f(x) \geq 0\}$

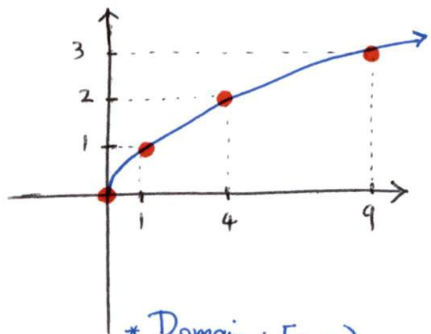
<u>Example</u> : $f(x) = \sqrt{x}$ $\leadsto$ Domain : $x \geq 0$

Let's sketch : $f(0) = \sqrt{0} = 0$, $f(4) = \sqrt{4} = 2$

$f(1) = \sqrt{1} = 1$, $f(9) = \sqrt{9} = 3$

* Note that we don't compute $\sqrt{-1}$, or $\sqrt{-4}$, ...

because square root is NOT defined for negative values.



* Domain : $[0, \infty)$

* Range : $[0, \infty)$

When different types of functions are combined, we find the domain of each part and combine the domain so that it works for the whole function.

<u>Examples</u>. Find the domain of the following functions.

(1) $f(x) = \underbrace{x^5 + 2x^3 - 3}_{\text{Domain} : \mathbb{R}} + \underbrace{\frac{1}{3x - 2}}_{\text{Domain} : \mathbb{R} - \{\frac{2}{3}\}}$ $\leadsto$ Combine these two

$3x - 2 = 0 \Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}$

$\Rightarrow$ Domain of $f(x) = \mathbb{R} - \{\frac{2}{3}\}$

(2) $g(x) = \underbrace{1}_{\boxed{\text{Domain}: \mathbb{R}}} + \sqrt{x+1} + \underbrace{3x}_{\boxed{\text{Domain}: \mathbb{R}}}$

$\rightsquigarrow$ takes any x-value returns 1.

combine these three

Domain : $\{x \in \mathbb{R} : x \geq 1\}$
$= [1, \infty)$

$\sqrt{x+1} \rightsquigarrow$ is defined when $x + 1 \geq 0 \Rightarrow \boxed{x \geq 1}$

(3) $f(x) = \dfrac{1}{\sqrt{6-2x}} + \dfrac{1}{1-x}$

A. $(-\infty, 3)$

B. $(3, \infty)$

C. $(-\infty, 1) \cup (1, 3]$

D. $(-\infty, 1) \cup (1, 3)$

E. $(-\infty, 1) \cup (3, \infty)$

$\dfrac{1}{\sqrt{6-2x}} \rightsquigarrow 6 - 2x > 0 \Rightarrow 6 > 2x$

$\Rightarrow \dfrac{6}{2} > x$

$\Rightarrow \boxed{3 > x} \to$ One part

* Note : $\sqrt{6-2x}$ is in the denominator, so it can NOT be $= 0$, that's why we have $6 - 2x > 0$ and NOT $6 - 2x \geq 0$

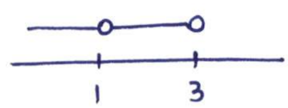
* Note : When you solve $6 - 2x > 0$, if you do

$-2x > -6 \xrightarrow{\div -2} x < \dfrac{-6}{-2} = 3$

inequality must change direction.

$\dfrac{1}{1-x} \rightsquigarrow \boxed{\text{Domain} : \mathbb{R} - \{1\}}$

Now combine the two parts : $x < 3$, $x \neq 1$ $\xrightarrow{\text{real line}}$



$\Rightarrow$ Interval Notation : $(-\infty, 1) \cup (1, 3)$ $\boxed{D}$

(4) $f(x) = \dfrac{(x+1)(x-3)}{(x-3)} \rightsquigarrow$ similar question will be done in lab.