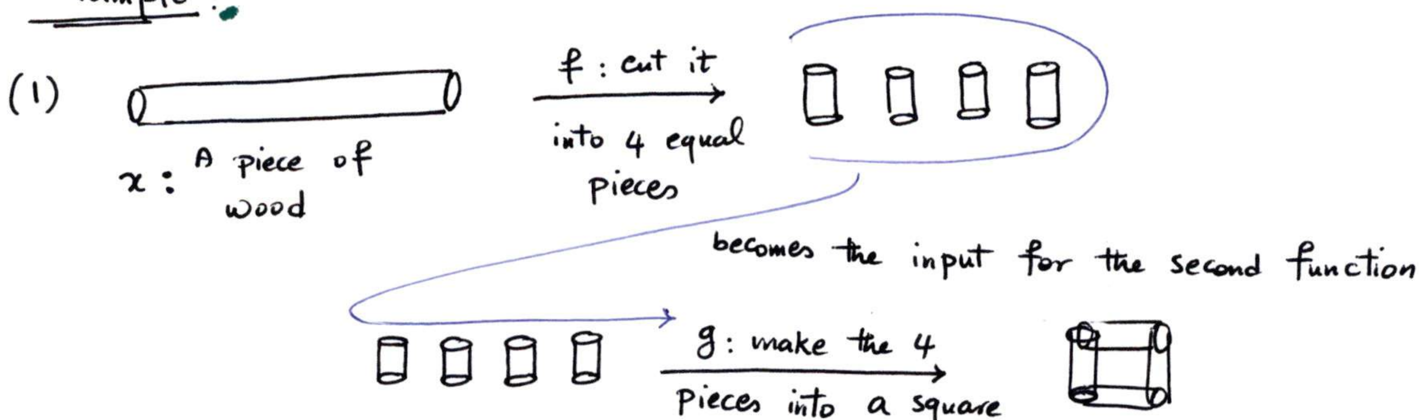
Composition of Functions: Given two functions, say $f(x)$ and $g(x)$; we can take the output from one function and use it as the input for the second function.

Example:

(1)



$$\underbrace{g\,(\underbrace{f(x)}_{1^{st}\text{ operation on } x})}_{2^{nd}\text{ operation on } f(x)}$$

(2) $f(x) = x + 1$

$g(x) = x^2$

$$x \xrightarrow[x+1]{f(x)=} x+1 \xrightarrow[x^2]{g(x)=} (x+1)^2$$

$$g\,(f(x)) = (x+1)^2$$

1^{st}: f applies on x

2^{nd}: g applies on $f(x)$.

Notation:

Function composition is denoted by "$\circ$", so

$$f(g(x)) = f \circ g(x)$$

$$g(f(x)) = g \circ f(x)$$

Question: Is $f \circ g(x) = g \circ f(x)$ in general? NO, check example 2 above:

$$g \circ f(x) = g(f(x)) = g(x+1) = (x+1)^2$$

$$f \circ g(x) = f(g(x)) = f(x^2) = x^2 + 1$$

$\neq$

Example. Let $f(x) = \dfrac{x}{x-3}$, $g(x) = \sqrt{1+x}$ and

$$h(x) = \begin{cases} 2x & x \leq 1 \\ 5-x & x > 1 \end{cases}$$

Evaluate

sits in f $\qquad g(x)=\sqrt{1+x} \qquad\qquad f(x) = \frac{x}{x-3}$

- $f \circ g(3) = f(\underbrace{g(3)}_{\text{sits in } g}) = f(\sqrt{1+3}) = f(\sqrt{4}) = f(2) = \dfrac{2}{2-3} = \dfrac{2}{-1} = -2$

- $g \circ f(4) = g(f(4)) = g\left(\dfrac{4}{4-3}\right) = g(4) = \sqrt{1+4} = \sqrt{5}$

$h(x) = 5-x$

- $h \circ f(5) = h(f(5)) = h\left(\dfrac{5}{5-3}\right) = h\left(\dfrac{5}{2}\right) = 5 - \dfrac{5}{2} = \dfrac{10-5}{2} = \dfrac{5}{2}$

$\frac{5}{2} > 1 \rightsquigarrow 2^{nd}$ line of h

$h(x) = 2x$

- $g \circ h(-1) = g(h(\underbrace{-1}_{-1 \leq 1 \rightsquigarrow 1^{st} \text{ line of } h})) = g(2.(-1)) = g(-2) = \sqrt{1+(-2)} = \sqrt{-1} \rightsquigarrow$ undefined

$\Rightarrow$ This implies that -1 is NOT in the domain of $g \circ h$

$h(x) = 5-x$

- $h \circ h(4) = h(h(\underbrace{4}_{4>1 \rightsquigarrow 2^{nd} \text{ line}})) = h(5-4) = h(\underbrace{1}_{1 \leq 1 \rightsquigarrow 1^{st} \text{ line}}) = 2.1 = 2$

$h(x) = 2x$

- $g \circ f(x) = g(f(x)) = g\left(\dfrac{x}{x-3}\right) = \sqrt{1 + \dfrac{x}{x-3}} = \sqrt{\dfrac{x-3+x}{x-3}} = \sqrt{\dfrac{2x-3}{x-3}}$

$g(x) = \sqrt{1+x}$

- $f \circ h(x) =$

* $h(x)$ is a piecewise function so to find $f(h(x))$ we need to consider the two intervals of the domain: $x \leq 1$ and $x > 1$.

* if $x \leq 1 \Rightarrow h(x) = 2x \rightsquigarrow f(h(x)) = f(2x) = \dfrac{2x}{2x-3}$

* if $x > 1 \Rightarrow h(x) = 5-x \rightsquigarrow f(h(x)) = f(5-x) = \dfrac{5-x}{(5-x)-3} = \dfrac{5-x}{2-x}$

so
$$f \circ h(x) = \begin{cases} \dfrac{2x}{2x-3} & x \leq 1 \\[2ex] \dfrac{5-x}{2-x} & x > 1 \end{cases}$$

* I did $g \circ h$ in class.

Example · Clicker

Let $f(x) = \begin{cases} 1-3x & x \geqslant 3 \\ -4+x & x < 3 \end{cases}$ and

$g(x) = \begin{cases} x^2+1 & x > -1 \\ -5x & x \leq -1 \end{cases}$. What is $f\circ g(-2)$?

A. 6 B. -14 C. -29 D. 1 E. NOT defined.

$$f\circ g(-2) = f(g(-2))$$

To find $g(-2)$ we use the 2^{nd} line of $g(x)$ because $-2 \leq -1$

so $\underset{g(x)=-5x}{g(-2) =} -5 \cdot -2 = 10$

Now $f(g(-2)) = f(10)$

since $10 \geqslant 3$, we use the 1^{st} line of f and plug 10 into

$1-3x \rightsquigarrow f(10) = 1 - 3\cdot(10) = -29$

$\Rightarrow \quad f\circ g(-2) = -29 \quad \boxed{C}$