Reminder:

- HW1 is due on Monday, Sept 14
- Office Hour

  Thursday: 11 - 12 in MATX 1118

  Friday: 2:30 - 3:30 in LSK 300

Today only: 2:45 - 3:45 in LSK 300

- MLC (Math Learning Center) is open now.

  Hours: Mon-Fri, 11 am - 5 pm

  Location: Rooms 301 and 302 in LSK

- Take a worksheet.

Trigonometry:

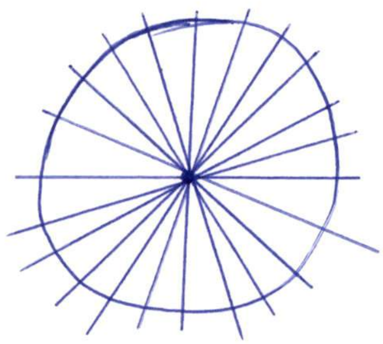
Lecture 5
Sept 14

Angles are an important part of trigonometric function, so let's start with a short review about angles.

Degree vs. Radian

Unlike numbers angles have a unit of measurement. There are two ways to read an angle: Degree & Radian.

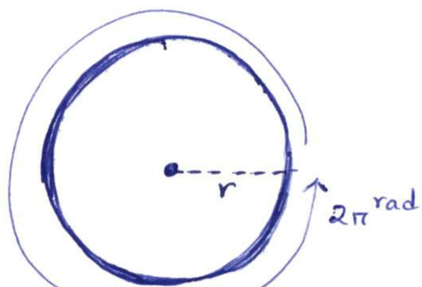
Degrees: Cut the circle into 360 equal pieces, each piece is called 1 degree, so



- A complete circle $= 360^\circ$
- Half circle $= 180^\circ$
- A quarter of a circle $= 90^\circ$

But the natural unit to use (and we use it in Calculus) is

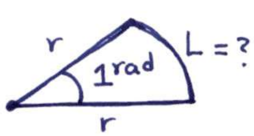
Radians: 2π radian corresponds to an entire circle



- Half a circle : π^{rad}
- A quarter of a circle : $\frac{\pi}{2} = \frac{2\pi}{4}$

Question. 1. What is the length of the arc of a complete circle? Its circumference $= 2\pi r$

2. What is the arc length corresponding to a slice of 1^{rad}.



1^{rad} is $\frac{1}{2\pi}$ of an entire circle so the arc length is $\frac{1}{2\pi}$ of the arc of the whole circle

$$\Rightarrow L = \frac{1}{2\pi} \cdot 2\pi r = r$$

⊛ 1^{rad} corresponds to a slice with arc length equal to 1.

Clicker Q. $\frac{2\pi}{3}$ rad = ? degrees

A. $135°$

B. $45°$

C. $60°$

D. $120°$

$\pi = 180°$, $2\pi = 360°$

$\Rightarrow \frac{2\pi}{3} = \frac{360}{3} = 120°$

$\Rightarrow \boxed{D}$

Convert degrees ⟷ radian

$135° \rightsquigarrow 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$

θ = angle in deg $\rightsquigarrow$ $\theta° \times \frac{\pi}{180} = \theta$ in rad

$\theta° = \theta^{rad} \times \frac{180}{\pi}$ $\leftsquigarrow$ θ in rad

$\frac{11\pi}{6}$ rad $\rightsquigarrow \frac{11\pi}{6} \times \frac{180}{\pi} = 330°$

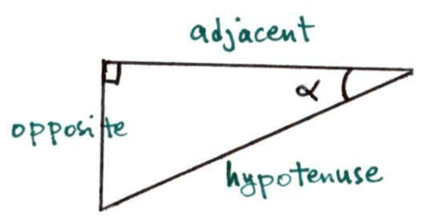
(*) Important to remember: know the following common angles in radian.

Degree	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

(*) Note that when there's NO unit is given for the angle, it's assumed that the unit is radian.

Trig Ratios

: They're all about a right triangle and the relation b/w the lengths of its sides and the angles.



$\cos\alpha = \frac{adj}{hyp}$

$\sin\alpha = \frac{opp}{hyp}$

other ratios →

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{opp}{adj}$

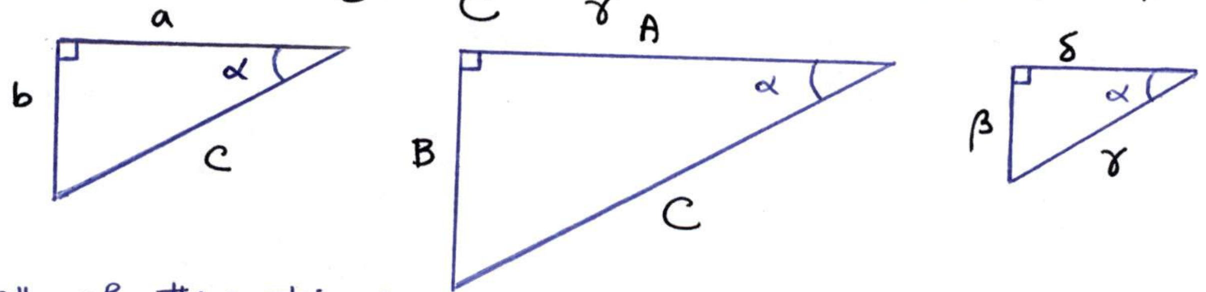
$\cot\alpha = \frac{1}{\tan\alpha} = \frac{\cos\alpha}{\sin\alpha}$

$\sec\alpha = \frac{1}{\cos\alpha}$

$\csc\alpha = \frac{1}{\sin\alpha}$

* adj and opp may change depending on where we choose angle α.

Remark: Note that if we change the size of the right triangle but keep the angle α unchanged, the trig ratios remain fixed: $\frac{b}{c} = \frac{B}{C} = \frac{\beta}{\gamma} = \sin\alpha$ and so on for all others.

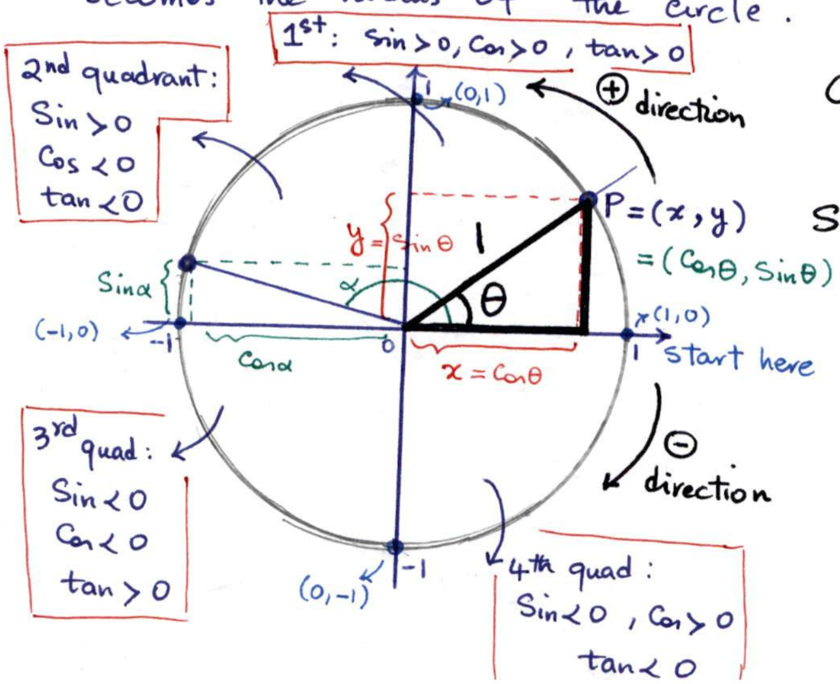


In all of these triangles, $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, ... are all the same even though the side lengths are changed.

Trigs most useful tool:

The Unit Circle

Consider a right triangle whose hypotenuse = 1, and also consider a circle centred at the origin with radius = 1. Place the triangle inside the circle such that its hypontenuse becomes the radius of the circle.



$$\left.\begin{aligned} \cos\theta &= \frac{adj}{hyp} = \frac{x}{1} \\ \sin\theta &= \frac{opp}{hyp} = \frac{y}{1} \end{aligned}\right] \Rightarrow \boxed{\begin{aligned} \cos\theta &= x \\ \sin\theta &= y \end{aligned}}$$

→ From the unit circle, we get:

$\boxed{-1 \le \cos\theta \le 1}$, $\boxed{-1 \le \sin\theta \le 1}$

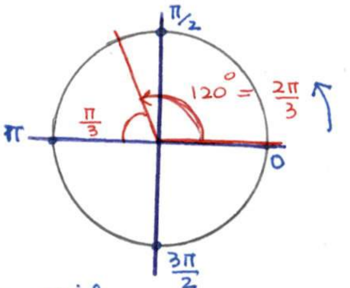
→ Rotating on the circle for more than one cycles, does NOT change sin or cos, just angles get bigger or smaller

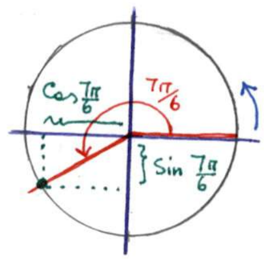
Read $\sin\theta$ and $\cos\theta$ in the unit circle:

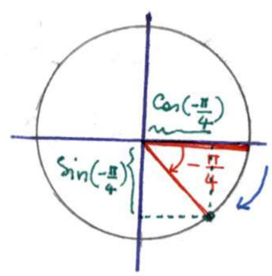
* The positive x-axis is where the angle starts.
* Go counter clock-wise for $\oplus$ angle and clock-wise for $\ominus$ angle.
* Find the radius of the circle corresponding to angle θ and the point P where the radius intersects the circle.
* The x-coordinate of $P = \cos\theta$
* The y-coordinate of $P = \sin\theta$

$$\Longrightarrow P = (x, y) = (\cos\theta, \sin\theta)$$

For example, let's find angles $\theta = \frac{2\pi}{3}, \frac{7\pi}{6}, -\frac{\pi}{4}$







$\theta = \frac{2\pi}{3} \xrightarrow{\text{which}} \text{quad?}$ $\quad \frac{360}{3} = 120$

Better way: Break the angle

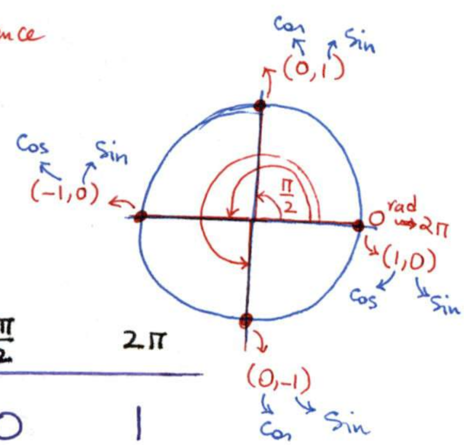
$$\frac{2\pi}{3} = \frac{3\pi - \pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3}$$

$$= \underbrace{\pi - \frac{\pi}{3}}_{2^{nd}\text{ quadrant}} \rightarrow \text{reference angle}$$

$$\theta = \frac{7\pi}{6} = \frac{6\pi + \pi}{6}$$

$$= \frac{6\pi}{6} + \frac{\pi}{6} = \underbrace{\pi + \frac{\pi}{6}}_{3^{rd}\text{ quad}} \rightarrow \text{reference}$$

$$\theta = -\left(\frac{\pi}{4}\right) \rightarrow \text{reference}$$



Sin/Cos of common angles:

x	0	$\pi/6$ → 30°	$\pi/3$ → 60°	$\pi/4$ → 45°	$\pi/2$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	0	-1	0	1
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	1	0	-1	0
$\frac{\sin x}{\cos x} = \tan x$	$\frac{0}{1} = 0$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\sqrt{3}$	1	$\frac{1}{0} = \infty$ =undefined	0	∞	

Graph of Sin and Cos :

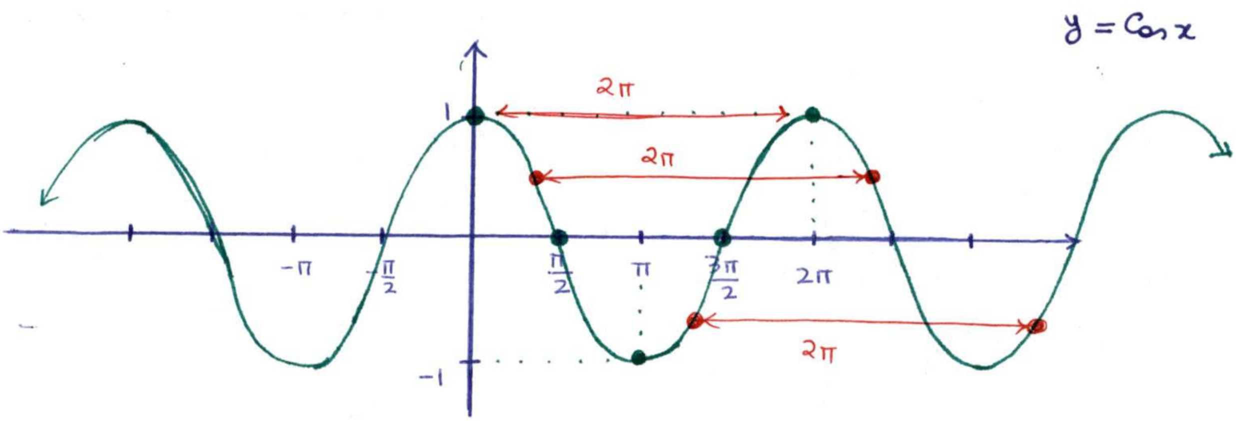
Take the angle as the input , then we can construct trig functions.

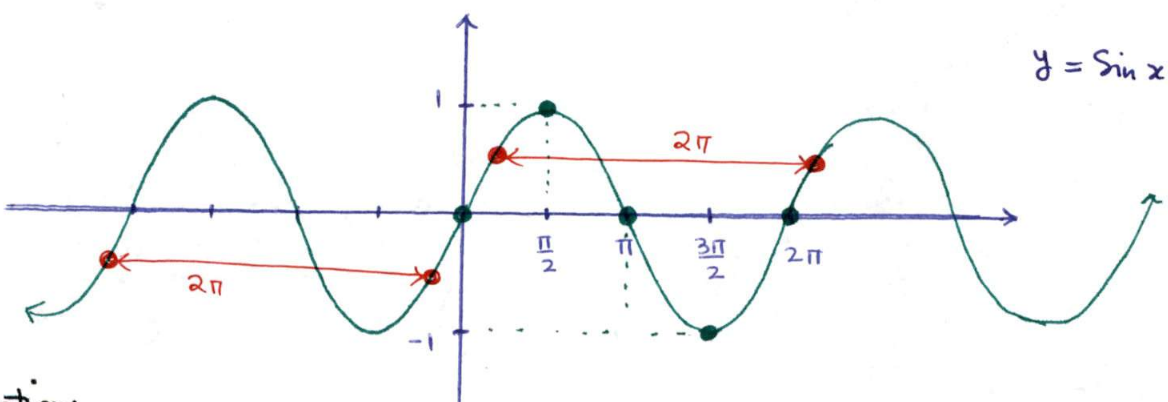
angles in radian $\begin{pmatrix} \theta \\ x \end{pmatrix}$ $\begin{matrix} \theta \xmapsto{\text{Sin}} \sin\theta \\ x \xmapsto{\text{Cos}} \cos x \end{matrix}\Bigg\}$ numbers angles in radian $\begin{pmatrix} \alpha \\ t \end{pmatrix}$ $\begin{matrix} \alpha \xmapsto{\tan} \tan\alpha \\ t \xmapsto{\sec} \sec t \end{matrix}\Bigg\}$ numbers ...

* whatever the notation is , we keep in mind that the input $(\theta, x, t, \cdots)$ is an angle in radian and output is . . . .

We can use the table (last page) to sketch the graph of

$f(x) = \sin x$ and $f(x) = \cos x$.





Observations

* Domain : $\mathbb{R}$ (for both)
* Range : $[-1, 1]$ $\rightsquigarrow$ Sin and Cos NEVER return a number greater than 1 or less than −1 .
* $\sin x$ and $\cos x$ repeat themselves every 2π .
  ↳ periodic functions with period $= 2\pi$ i.e. $\sin(\theta) = \sin(\theta + 2\pi)$
  $\cos(\theta) = \cos(\theta + 2\pi)$

Clicker Q: Evaluate $\sin\left(\frac{4\pi}{3}\right)$ and $\cos\left(\frac{4\pi}{3}\right)$ (respectively)

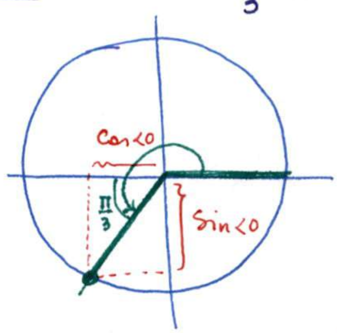
A. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$

C. $-\frac{1}{2}$, $\frac{\sqrt{3}}{2}$

B. $-\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$

D. $-\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$

1st step: Locate $\frac{4\pi}{3}$ in the unit circle.



$$\theta = \frac{4\pi}{3} = \frac{3\pi+\pi}{3} = \pi + \underbrace{\frac{\pi}{3}}_{3^{rd}\text{ quad}}$$ reference

Table:

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

3rd quad $\Rightarrow$ $\sin < 0$, $\cos < 0$

$$\Longrightarrow \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \Rightarrow \boxed{B}$$

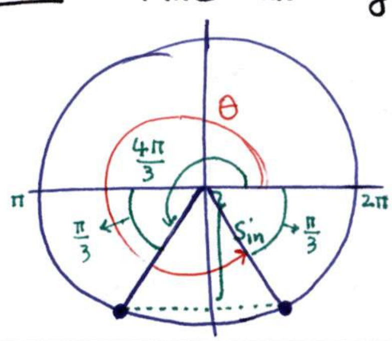
$$\cos\frac{4\pi}{3} = -\frac{1}{2}$$

* To find Sin and Cos:

- quadrant gives the sign positive/negative x or y
- Use the table for the numbers of sin and cos of the reference angle.

Question 1. Find an angle θ for which $\sin\theta = \sin\frac{4\pi}{3}$.

Question 2. Find an angle α for which $\cos\alpha = \cos\frac{4\pi}{3}$.

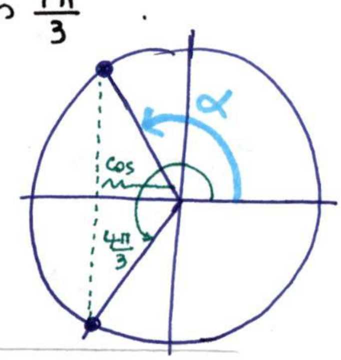


* The angle in red has the same Sin value as $4\pi/3$

$$\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi - \pi}{3} = \frac{5\pi}{3}$$

* The blue angle has the same Cos value as $4\pi/3$

$$\alpha = \pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$$



Practice Questions: Find Sin, Cos, tan and sec of the given angles.

(a) $-\frac{\pi}{3}$

(b) $\frac{7\pi}{3}$

(c) $\frac{18\pi}{2}$

(d) $\frac{19\pi}{6}$

(e) $\frac{5\pi}{6}$

(f) $-\frac{2\pi}{3}$

(g) $-\frac{5\pi}{4}$