Exponents

Suppose "a" is a real number and "n" is a positive integer, we then have :

$$\underset{\text{base}}{a}^{\overset{\text{exponent or power}}{n}} = \underbrace{a \cdot a \cdot \ \cdots \ \cdot a}_{n \text{ times}}$$

Exponent Laws :

- $$a^n \cdot a^m = \underbrace{a \cdot \ \cdots \ \cdot a}_{n \text{ times}} \cdot \underbrace{a \cdot \ \cdots \ a}_{m \text{ times}} \Rightarrow \boxed{a^n \cdot a^m = a^{n+m}}$$
  $$= \underbrace{a \cdot a \cdot \ \cdots \ \cdot a}_{n+m \text{ times}} = a^{n+m}$$

- $a^1 = a$

- $a^{-1} = \dfrac{1}{a}$

- $$a^n \cdot b^n = \underbrace{a \cdot \ \cdots \ \cdot a}_{n \text{ times}} \cdot \underbrace{b \cdot \ \cdots \ \cdot b}_{n \text{ times}} \Rightarrow \boxed{a^n \cdot b^n = (ab)^n}$$
  $$\overset{\text{pair}}{=} \underbrace{ab \cdot ab \cdot \ \cdots \ \cdot ab}_{n \text{ times}} = (ab)^n$$

- $$(a^n)^m = \underbrace{a^n \cdot \ \cdots \ \cdot a^n}_{m \text{ times}} = (\underbrace{a \cdot \cdots \cdot a}_{n \text{ times}}) \cdot \ \cdots \ \cdot (\underbrace{a \cdot \cdots \cdot a}_{n \text{ times}})$$
  $$= \underbrace{a \cdot \ \cdots \ \cdot a \cdot \ \cdots \ \cdot a \cdot \cdots \cdot a}_{mn \text{ times}} = a^{mn}.$$

- $a^{-n} = (a^n)^{-1} = \dfrac{1}{a^n}$

- What is a^0 ? Write $a^0 = a^{2-2} = a^2 \cdot a^{-2} = a^2 \cdot \dfrac{1}{a^2} = 1$

* Note : 0^0 is special. $0^0 \neq 1$

Laws of Exponents (Review)

Law	Example
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = \frac{1}{x}$	$4^{-1} = \frac{1}{4}$
$x^m \cdot x^n = x^{m+n}$	$x^2 \cdot x^3 = x^{2+3} = x^5$
$x^m / x^n = x^{m-n}$	$x^6 / x^2 = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2 \cdot 3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
$\left(x/y\right)^n = \frac{x^n}{y^n}$	$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$
$x^{-n} = \frac{1}{x^n}$	$x^{-3} = \frac{1}{x^3}$
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$x^{2/3} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2$

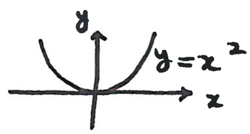
Making functions with exponents ?

take the input variable, say x, depending on where x sits, two type of functions are produced

power functions like : $x^2, x^3, x^{\frac{1}{2}}$

Exponential functions like $2^x, 3^x, \left(\frac{1}{2}\right)^x$

* These are two very different family of functions. For example, we know $f(x) = x^2$ is a quadratic function whose graph is a parabola



, but $f(x) = 2^x$ is an exponential function with a different graph.

Question. Sketch the graph of $f(x) = 2^x$ and find its domain, range and intercepts.

* To sketch the graph we need a table of values, so let's make one.

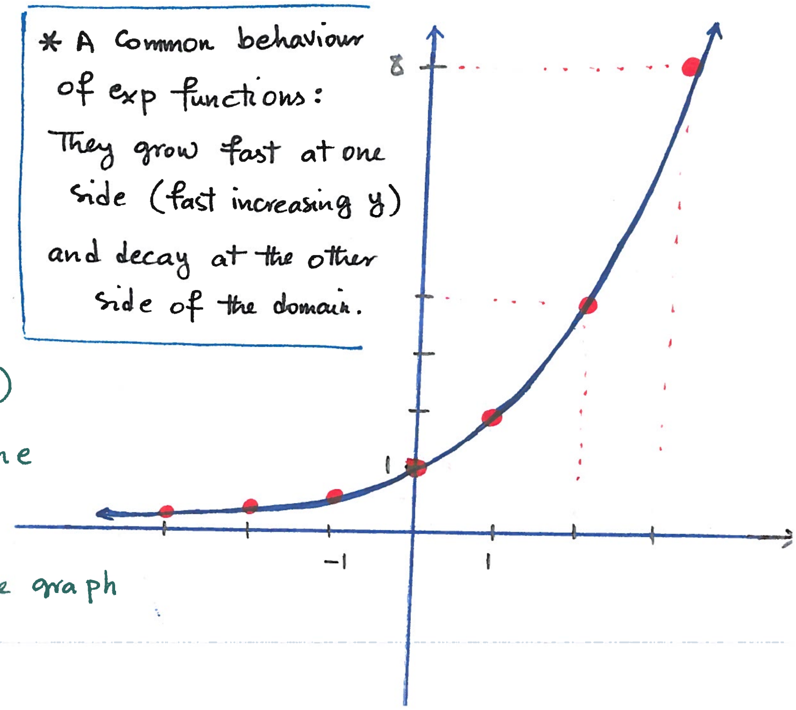
x	$y = 2^x$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$

→ output is always positive.

* Domain : $\mathbb{R}$
* Range : $(0, \infty)$
* x-intercept : None
* y-int : 1

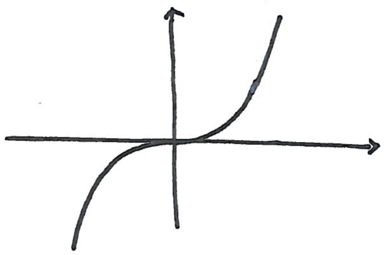
↳ $(0,1)$ on the graph

* A common behaviour of exp functions: They grow fast at one side (fast increasing y) and decay at the other side of the domain.



Clicker Q : Which one is the graph of $f(x) = 3^x$?

A.



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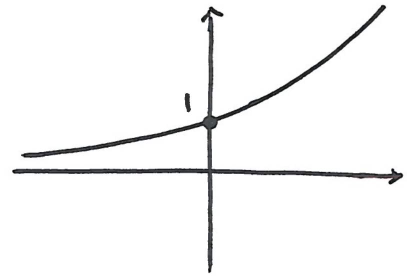
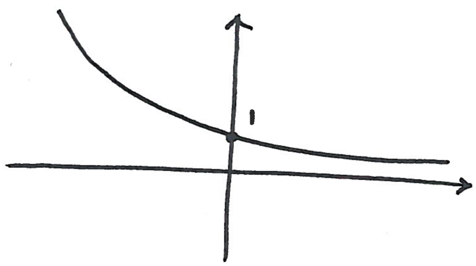
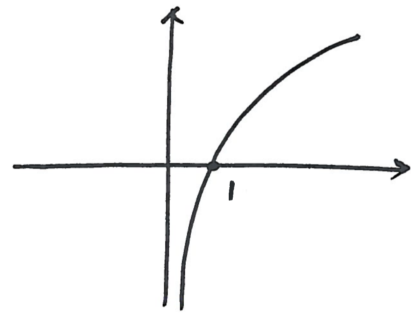


C.



D.



What about $f(x) = \left(\frac{1}{3}\right)^x$?

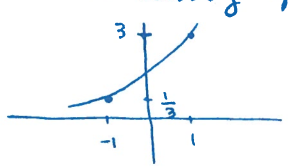
* $\left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x}$

* $f(x) = 3^x$: Check some points and we eliminate the options :

y-int $\xrightarrow{x=0}$ $3^0 = 1$ ⇝ Only B and C

$f(1) = 3^1 = 3$

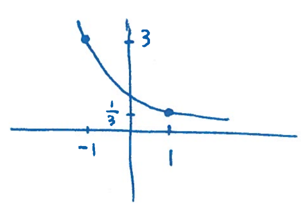
$f(-1) = 3^{-1} = \frac{1}{3}$ ⇝ Increasing graph ⇝ $\boxed{B}$



* $f(x) = \left(\frac{1}{3}\right)^x$

$f(1) = \left(\frac{1}{3}\right)^1 = \frac{1}{3}$

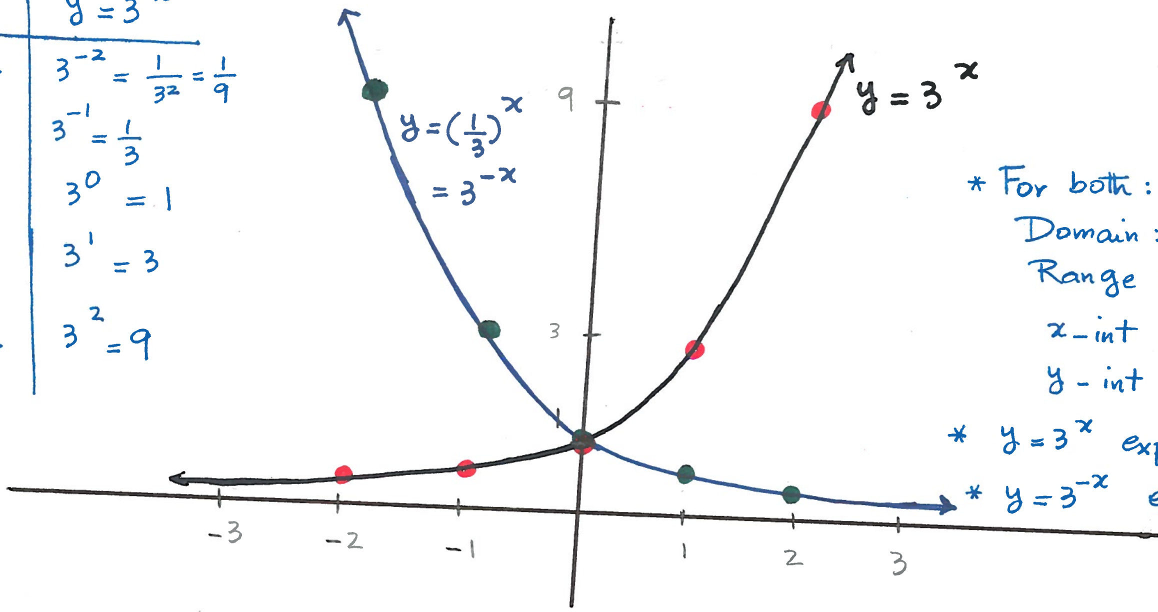
$f(-1) = \left(\frac{1}{3}\right)^{-1} = \frac{1}{3^{-1}} = 3$



⇝ Decreasing graph $\boxed{C}$

* How does graph of $f(x) = 3^x$ look like ?

$y=(\frac{1}{3})^x$	x	$y=3^x$
9	-2	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
3	-1	$3^{-1} = \frac{1}{3}$
0	0	$3^0 = 1$
$\frac{1}{3}$	1	$3^1 = 3$
$\frac{1}{9}$	2	$3^2 = 9$



* For both :

Domain : $\mathbb{R}$

Range : $(0, \infty)$

x-int : None

y-int : 1

* $y = 3^x$ exponential growth

* $y = 3^{-x}$ exponential decay

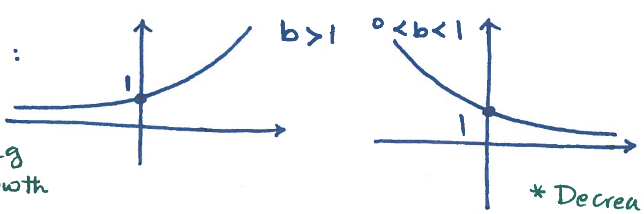
* What about $f(x) = (\frac{1}{3})^x = 3^{-x}$? Very similar behaviour in terms of domain and range, but due to the sign change in the exponent, all y-values are flipped so the graph becomes decreasing.

Definition (Exponential Function)

$$f(x) = b^x \quad \text{for} \quad b > 0, \; b \neq 1$$

Observations :

Graph :



* Increasing ⇝ Growth

* Decreasing ⇝ Decay

• y-intercept $= 1$

Domain : $(-\infty, \infty)$

Range : $(0, \infty)$ ⇝ No x-intercepts.

0 not included because the graph NEVER touches the x-axis. This means $b^x = 0$ NEVER happens.

⇝ Exponential function always produces positive values.

For example : $2^x = -1$, $(\frac{1}{2})^x = -3$, $3^x = 0$ NEVER have solutions.

Very Important exponential function:

$\boxed{f(x) = e^x}$ ⤳ Natural exponential function.

This function appears in many natural phenomena, and in general we just call it the exponential function.

⤳ What is "e"? $2 < e < 3$

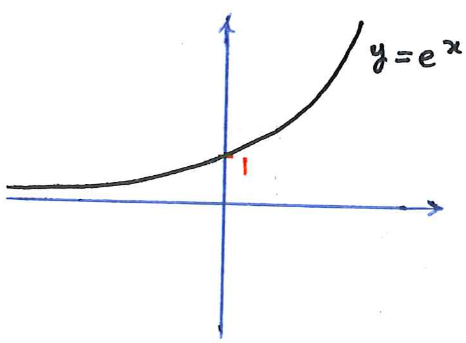
"e" is a math constant, and it is approximately

$$e = 2.718281828 45905 \ldots$$

like π; e is also an irrational number, and there are different ways to define this number. We'll see one of them later.

⊛ One of the important applications of $y = e^x$ is in the study of "compound interest".

⊛ All the properties of $y = b^x$ applies to $y = e^x$.



✓ Domain = $(-\infty, \infty)$

✓ Range = $(0, +\infty)$ ⤳ Always positive

✓ y-int = 1

✓ NO x-int ⤳ Does NOT cross the x-axis.

Example. Simplify

$$\frac{(\sqrt{e}\cdot e^{2})^{3}}{e} = \frac{\left(e^{\frac{1}{2}}\cdot e^{2}\right)^{3}}{e}$$

$$= \frac{\left(e^{\frac{1}{2}+2}\right)^{3}}{e}$$

or

$$= \frac{e^{\frac{3}{2}}\cdot e^{6}}{e}$$

$$= \frac{\left(e^{\frac{5}{2}}\right)^{3}}{e}$$

$$= \frac{e^{\frac{3}{2}+6}}{e}$$

$$= \frac{e^{\frac{15}{2}}}{e}$$

$$= e^{15/2 - 1}$$

$$= e^{\frac{13}{2}}$$