Clicker Q : Simplify $\dfrac{(2^x)^3 \cdot (6^x)^4 \cdot 2^x}{3^{5x} \cdot 2^{10x}}$

A. 6^{-x}

C. 6^{4x}

B. 12^x

D. 12^{-x}

Start with the top terms

$\boxed{a^n \cdot a^m = a^{n+m}}$

$* \ (a^n)^m = a^{nm} \quad \Rightarrow \quad \dfrac{2^{3x} \cdot 6^{4x} \cdot 2^x}{3^{5x} \cdot 2^{10x}} = \dfrac{2^{3x+x} \cdot 6^{4x}}{3^{5x} \cdot 2^{10x}}$

$$= \frac{2^{4x} \cdot 6^{4x}}{3^{5x} \cdot 2^{10x}}$$

$* \ a^m \cdot b^m = (ab)^m$

$$= \frac{12^{4x}}{3^{5x} \cdot 2^{10x}}$$

How to simplify the denominator ?

Not equal base, NOT equal exponent

But

$2^{10x} = (2^2)^{5x} = 4^{5x}$

$$= \frac{12^{4x}}{3^{5x} \cdot 4^{5x}}$$

$$= \frac{12^{4x}}{12^{5x}}$$

$* \ \dfrac{a^n}{a^m} = a^{n-m}$

$= 12^{4x-5x} = 12^{-x} \rightsquigarrow \boxed{D}$

Logarithmic Functions :

Suppose we'd like to solve $2^x = 16 \rightsquigarrow x = \ldots$

But what if $e^x = 16$, then $x = ?$

We need logarithm

$x = \log_e 16$

What is logarithm ?

Closely related to exponentials :

$$2^x = 16 \xrightarrow[\text{representation}]{\text{another}} \log_2 16 = 4$$

In general $b^x = y \rightsquigarrow \log_b y = x$

Now, we can consider a logarithmic function in a base $b>0$ $(b \neq 1)$ to be

$$x \xrightarrow{\log_b} y = \log_b x$$

$f(x) = \log_b x$

Example . Find y :

- $\log_3 9 = y \xrightarrow[\text{to exp}]{\text{convert}} 3^y = 9 \Rightarrow \boxed{y=2}$
- $\log_{10} y = 0 \rightsquigarrow 10^0 = y \Rightarrow \boxed{y=1}$
- $\log_{17} 1 = y \rightsquigarrow 17^y = 1 \Rightarrow \boxed{y=0}$
- $\log_5 y = 1 \rightsquigarrow 5^1 = y \Rightarrow \boxed{y=5}$

* log has important physical applications :

- Earthquack intensity and Richter scale
- Sound intensity
- Stars brightness

⋮

We see that if $b^x = y$

$\xrightarrow[\text{we take } \log_b]{\text{to find } x}$ $x = \log_b y$ $\rightsquigarrow$ log undo exp and vice versa

$\Rightarrow$ What is the relation between

$$f(x) = b^x \quad \text{and} \quad g(x) = \log_b x \quad ?$$

b^x and $\log_b x$ are inverse of each other.

One of them makes the other one undo.

$$x \xrightarrow{f(x) = b^x} b^x \xrightarrow{g(x) = \log_b x} \boxed{\log_b b^x = x}$$ $\rightarrow$ Remember this rule!

* Start with x $\rightsquigarrow$ find b^x $\rightsquigarrow$ take log of b^x $\rightsquigarrow$ operation is undone $\rightsquigarrow$ back to x

$$g(f(x)) = x$$

$$x \xrightarrow{g(x) = \log_b x} \log_b x \xrightarrow{f(x) = b^x} \boxed{\log_b b^x = x}$$ $\rightarrow$ Remember this rule!

$$f(g(x)) = x$$

$\Rightarrow$ What other functions do we know that are inverse of each other?

- $f(x) = x^2$, $g(x) = \sqrt{x}$

  $2 \xrightarrow[f]{x^2} 4 \xrightarrow[g]{\sqrt{x}} \sqrt{4} = 2 \rightsquigarrow \quad g\circ f(2) = 2$

  $9 \xrightarrow[g]{\sqrt{x}} \sqrt{9} = 3 \xrightarrow{x^2} 3^2 = 9 \rightsquigarrow \quad f\circ g(3) = 3$

  $\rightsquigarrow \boxed{\begin{array}{l} f\circ g(x) = x \\ g\circ f(x) = x \end{array}}$

  This is property of inverse functions.

- In general x^n and $\sqrt[n]{x}$ are inverse of each other. x^3 & $\sqrt[3]{x}$, x^4 & $\sqrt[4]{x}$, ...

- What about $f(x) = x$ and $g(x) = \frac{1}{x}$?

  Let's verify: $f\circ g(x) = f(\frac{1}{x}) = \frac{1}{x}$ ✗

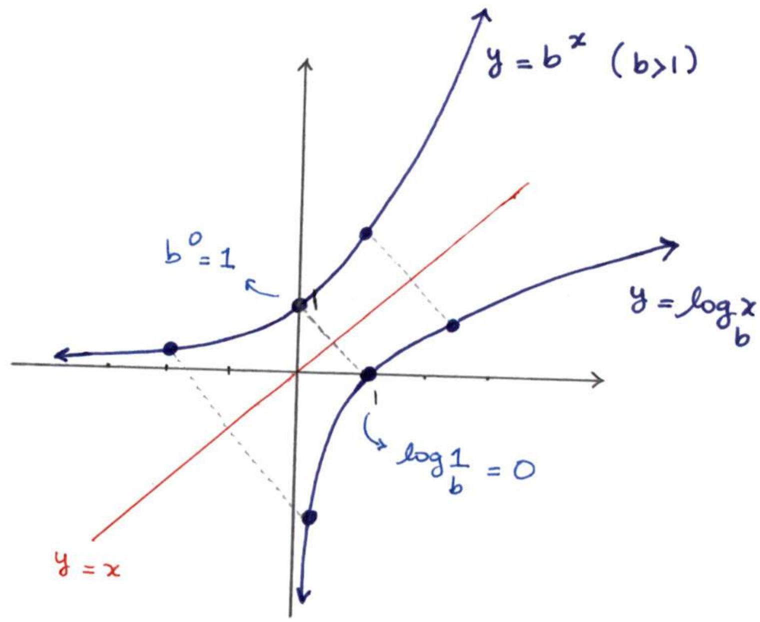
  NO.

  $g\circ f(x) = g(x) = \frac{1}{x}$ ✗

Graph of $y = \log_b x$ $(b>0)$

* Graphical relation between two inverse functions:
  f and g inverse of each other, to find graph of g reflect graph of f across the line $\underbrace{y = x}$.

the line that divides 1^{st} and 3^{rd} quadrant into halves.



Properties :

Domain : $(0, \infty)$ ⟶ NOT allowed to input 0 and negative numbers for $\log_b x$
⟶ $\log_b 0$ = undefined, $\log_b -2$ = undefined,

Range : $(-\infty, \infty)$ ⟶ the output of $\log_b x = y$ can be any real number.

NO y-intercept ⟶ graph never crosses the y-axis.

x-int $= 1$

Compare with $y = b^x$:

Since $b^x = y$ and $\log_b x = y$ are inverse of each other

Domain of b^x = Range of $\log_b x$

Range of b^x = Domain of $\log_b x$

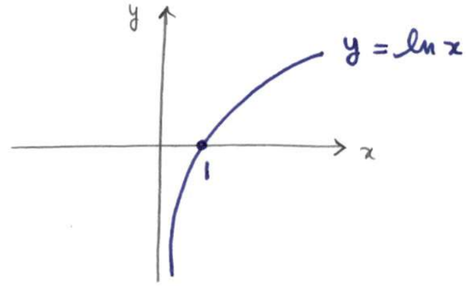
y-int of b^x = x-int of $\log_b x$

x-int of b^x = y-int of $\log_b x$

Especially important case :

$b = e \rightsquigarrow \log_e x = \ln x \rightsquigarrow$ when the log base is e, we remove the base and change log to ln.

* ln x is sometimes called natural logarithm.



* $y = e^x$ and $y = \ln x$ are inverse functions

i.e. $\boxed{\ln(e^x) = x}$ & $\boxed{e^{\ln x} = x}$

Laws of Logarithm (b > 1)

- $\boxed{\log_b b = 1 \xrightarrow{b=e} \ln e = 1}$
- $\boxed{\log_b 1 = 0 \xrightarrow{b=e} \ln 1 = 0}$
- $\boxed{\log_b(xy) = \log_b x + \log_b y \xrightarrow{b=e} \ln(xy) = \ln x + \ln y}$

  Common mistakes : $\log_b xy = \log_b x \cdot \log_b y$ or $\log_b x+y = \log_b x + \log_b y$

  or $\log_b x+y = \log_b x \cdot \log_b y$
- $\boxed{\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \xrightarrow{b=e} \ln\left(\frac{x}{y}\right) = \ln x - \ln y}$

  Common mistakes: $\log_b\left(\frac{x}{y}\right) = \log_b x / \log_b y$ or $\log_b x-y = \log_b x - \log_b y$
- $\boxed{\log_b(x^n) = n \log_b x \xrightarrow{b=e} \ln x^n = n \ln x}$

Examples. Simplify (a) $\log_{10} 5 + \log_{10} 2 =$

(b) $\ln(e^2) - \ln \sqrt[5]{e^2} = 2 \ln e - \frac{2}{5} \ln e = 2 - \frac{2}{5} = \frac{8}{5}$

$e^{\frac{2}{5}}$

* $\ln e = 1$

Worksheet Practice: Logarithms

1. Evaluate the following logarithms.

a) $\log_2 16 =$

b) $\log_7 1 =$

c) $\ln \sqrt[3]{e} =$

d) $\log_{\frac{2}{3}} 27/8 =$

e) $\log_{10} 10 =$

f) $\ln 1 =$

2. Use properties of logs and write the following logs in expanded form.

a) $\log_2 \left(\frac{2ab}{c^3} \right) =$

b) $\ln \left[(x-4)(2x+5) \right]^2 =$

c) $\ln \left[(x-4)(2x+5)^2 \right] =$

d) $\log_8 \sqrt{xy} =$

3. Express each of the following as a single log.

a) $3\log_5 x + \log_5 y - 2\log_5 W =$

b) $\frac{1}{2} \left[(2\ln a + \ln b) - 5 \ln c \right] =$

c) $\frac{1}{2} \ln x - \frac{1}{3} \ln y =$

4. Solve the following equations:

a) $\log_2(x+1) + \log_2 3 = 1$

b) $\log_2(x+3) + \log_2 x = 2$

c) $\ln(x-4) + \ln x = \ln 21$

d) $4\ln(3x) = 4$

e) $\ln x = \ln(1-x)$

f) $7 + 2\ln x = 6$

g) $\log x + \log(x-1) = \log(3x+12)$

h) $\log_2(x^2-6x) = 3 + \log_2(1-x)$

i) $5 + e^{x+1} = 20$

j) $2^x = 7$

k) $4^{x-3} = 9$

---

5) Simplify each log.

a) $\ln e^5 =$

b) $e^{2\ln 5} =$

c) $10^{2+\log 5} =$

d) $\dfrac{\log 100}{\log 10} =$

e) $\dfrac{\log_3 9}{\log_2 8} =$