HW 3 . Solution :

(1)

(a) $$\lim_{x\to 1} \frac{\sqrt{x^2+2}-\sqrt{x+2}}{x-1} \cdot \frac{\sqrt{x^2+2}+\sqrt{x+2}}{\sqrt{x^2+2}+\sqrt{x+2}}$$

$$= \lim_{x\to 1} \frac{x^2+2-(x+2)}{(x-1)(\sqrt{x^2+2}+\sqrt{x+2})} = \lim_{x\to 1} \frac{x^2-x}{(x-1)(\sqrt{x^2+2}+\sqrt{x+2})} = \overset{\text{still}}{\frac{0}{0}}$$

$$= \lim_{x\to 1} \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(\,\dots)}$$

$$= \frac{1}{\sqrt{1+2}+\sqrt{1+2}} = \frac{1}{2\sqrt{3}}$$

(b) $$\lim_{x\to 3} \frac{x^2-9}{|x-3|} \longrightarrow \lim_{\underset{x\geq 3}{x\to 3^+}} \frac{x^2-9}{|x-3|} = \lim_{x\to 3^+} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = 3+3=6$$

$$* \; |x-3| = \begin{cases} x-3 & x\geq 3 \\ -(x-3) & x<3 \end{cases} \qquad \longrightarrow \lim_{x\to 3^-} \frac{x^2-9}{|x-3|} = \lim_{x\to 3^-} \frac{\cancel{(x-3)}(x+3)}{-\cancel{(x-3)}} = -(3+3)=-6$$

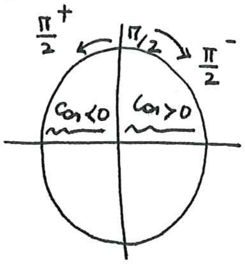
$$\Rightarrow \lim_{x\to 3} \frac{x^2-9}{|x-3|} = \text{DNE}$$

2. $$f(x) = \tan x = \frac{\sin x}{\cos x}$$

Candidates for V.A. are $x$ values at which $\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

$\Rightarrow x = (2n+1)\frac{\pi}{2} \quad n=0,\pm 1, \pm 2, \dots$

Verify the limits at $x=\frac{\pi}{2}$ :



$$\lim_{x \to \frac{\pi}{2}^+} \frac{\sin x}{\cos x} = \frac{1}{0^-} = -\infty$$

2nd ↙ quad   ↙ 2nd quadrant

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \frac{1}{0^+} = \infty$$

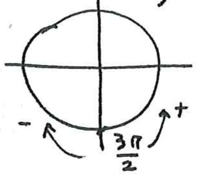
1st ↙   ↙ 1st quadrant

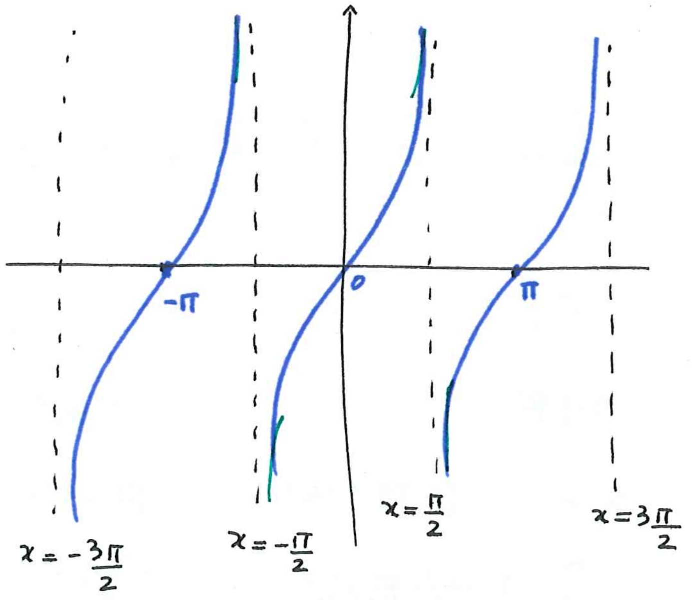
$$\lim_{x \to \frac{3\pi}{2}^+} \frac{\sin x}{\cos x} = \frac{-1}{0^+} = -\infty$$

4th ↙   4th ↙ quad

$$\lim_{x \to \frac{3\pi}{2}^-} \frac{\sin x}{\cos x} = \frac{-1}{0^-} = +\infty$$

3rd ↙   $0^-$ → 3rd quad





Similar pattern for all other V.A.

(3) $g(x) = e^{-2x} \sin(2x)$

(a) Domain : $\mathbb{R}$

(b) NO V.A. because the function is defined everywhere.

(c) As $x \to \infty$ : $\lim_{x \to \infty} e^{-2x} = 0$ ( $e^{-2x}$ gets close to 0 )

(d) As $x \to \infty$ : $\sin(2x)$ always remains between -1 and 1 : oscillating but it does NOT approach to a fixed value.

(e) As $x \to \infty$ : $e^{-2x} \sin(2x)$ approaches 0 so $y=0$ is a H.A. of g.

(4) $f(x) = \dfrac{3x+1}{x-1}$ , $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\overset{f(x+h)}{\boxed{\frac{3(x+h)+1}{x+h-1}}} - \overset{f(x)}{\boxed{\frac{3x+1}{x-1}}}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(3x+3h+1)(x-1) - (3x+1)(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{\cancel{3x^2} + \cancel{3hx} + \cancel{x} - \cancel{3x} - 3h - \cancel{1} - \cancel{3x^2} - \cancel{3xh} + \cancel{3x} - \cancel{x} - h + \cancel{1}}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{-4\cancel{h}}{(x+h-1)(x-1)} \cdot \frac{1}{\cancel{h}} = \lim_{h \to 0} \frac{-4}{(x+h-1)(x-1)}$$

$$\underset{\text{sub } h=0}{=} \frac{-4}{(x-1)^2}$$

(b)

tangent line is horizontal when $m_{tan} = 0$ so $f'(x) = 0$

$$\Rightarrow \frac{-4}{(x-1)^2} = 0 \xrightarrow{top=0} -4 = 0 \text{ NEVER}$$

$\Rightarrow$ For NO $x$, $m_{tan} = 0$

$\Rightarrow$ There is NO horizontal tangent line to $f(x)$.

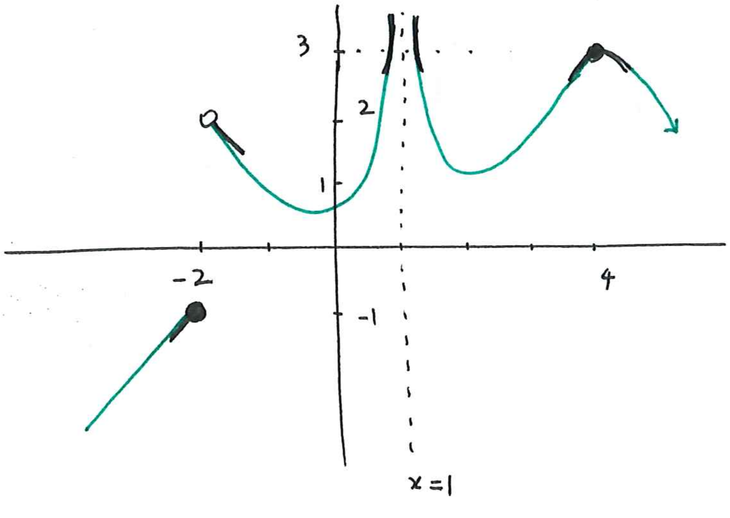
(c) $f(x)$ is always decreasing because

$$f'(x) = \frac{\overset{\nearrow \text{negative}}{-4}}{\underset{\searrow \text{positive}}{(x-1)^2}} < 0$$

always tangent line with negative slope.

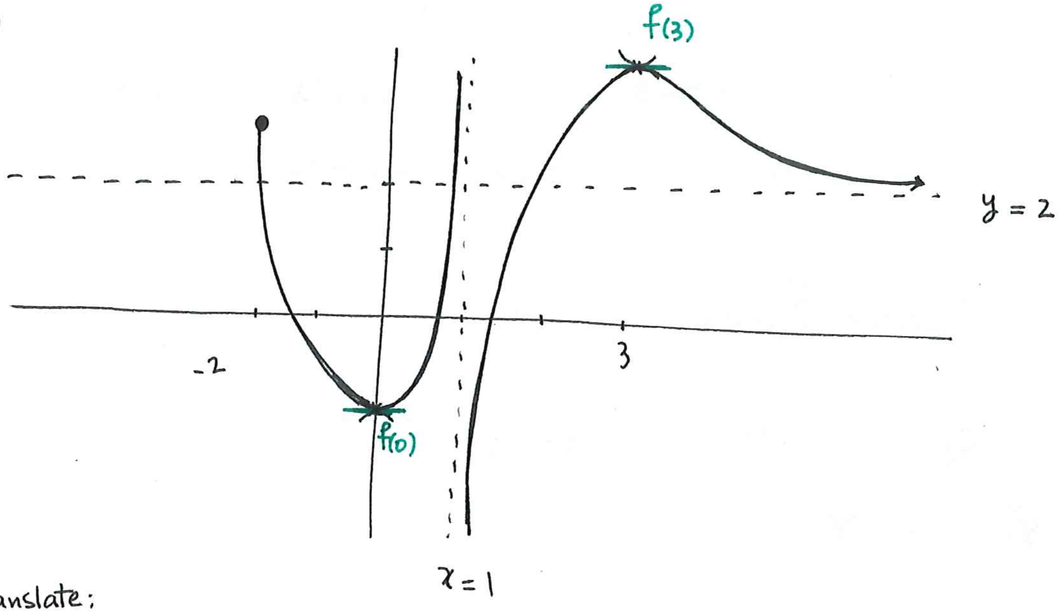
(5)

(a)



* Make small pieces of graph for each given condition and connect them eventually.

(b)



Translate:

* At $x=0$ and $x=3$ : Horizontal tangent line ⌒ or ∪

↳ we decide upward or downward bump based on the increasing and decreasing conditions.

* in $(-2,0)$ and $(3, \infty)$ decreasing $(f'<0)$

* in $(0,1)$ and $(1,3)$ increasing $(f'>0)$

* Note if you choose f(3) below the horizontal asymptote, the conditions will be violated.