Factoring Review:

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$a^2 - b^2 = (a-b)(a+b)$

Lecture 11
Sept 28

<u>Clicker Q</u> : Let $f(x) = \begin{cases} x+3 & x \geq 2 \\ -x^2-1 & x < 2 \end{cases}$. Choose the correct value for each of the following limits.

A. 5 B. −5 C. 2 D. DNE E. A & B

(1) $\lim\limits_{\boxed{x \to 2^-}} f(x) = \lim\limits_{x \to 2^-} -x^2 - 1 = -2^2 - 1 = -5 \rightsquigarrow \boxed{B}$

$x < 2 \rightsquigarrow 2^{nd}$ line of f

(2) $\lim\limits_{\boxed{x \to 2^+}} f(x) = \lim\limits_{x \to 2} x+3 = 2+3 = 5 \rightsquigarrow \boxed{A}$

$x > 2 \rightsquigarrow 1^{st}$ line of f

(3) $\lim\limits_{x \to 2} f(x) = \text{DNE}$ because $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$ so the full limit DNE. $\boxed{D}$

(4) $\lim\limits_{\boxed{x \to 3}} f(x) = \lim\limits_{x \to 3} x+3 = 3+3 = 6 \rightsquigarrow$ NO option ☹

$x > 2$ → approach 3 from left $\rightsquigarrow x \quad 2$

→ or approach 3 from right $\rightsquigarrow x > 2$

* For a piece-wise function, we need to be careful about the "border" x-values where the function breaks into multiple lines. ($x=2$ in this examp[le]) For other values than the "border" point we usually do not need to evaluate the one-sided limits separately.

Examples :

start with substitution

$$(1)\quad \lim_{x \to -4} \frac{x+4}{\frac{1}{4}+\frac{1}{x}} = \frac{-4+4}{\frac{1}{4}+\frac{1}{-4}} = \frac{0}{0} \quad !!!$$

Find the common denominator of the bottom fractions :

$$\lim_{x \to -4} \frac{x+4}{\frac{1}{4}+\frac{1}{x}} = \lim_{x \to -4} \frac{x+4}{\frac{x+4}{4x}} = \lim_{x \to -4} (x+4) \div \frac{x+4}{4x}$$

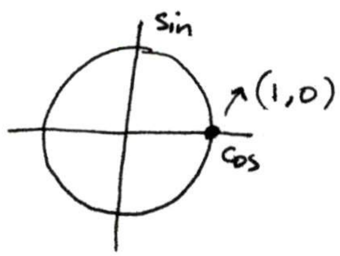
$$= \lim_{x \to -4} (\cancel{x+4}) \cdot \frac{4x}{\cancel{x+4}} = 4 \cdot (-4) = -16$$

problem found : $x+4$

re-substitute

substitute

$$(2)\quad \lim_{x \to 0} \frac{\sin x \cdot \cos x}{e^x} = \frac{\sin 0 \cdot \cos 0}{e^0}$$

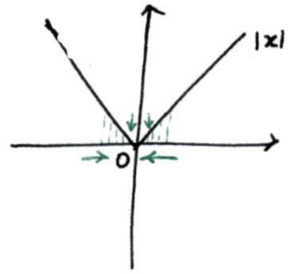


$$= \frac{0 \cdot 1}{1} = \frac{0}{1} = 0 \quad \ddot\smile$$

substitution worked .

(3) $\lim_{x \to 0} |x| = |0| = 0$

Substitute



(4) $\lim_{x \to 0} \frac{|x|}{x} = \frac{|0|}{0} = \frac{0}{0}$!!!

How to simplify ? Break $|x|$ into pieces.

Recall : $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

if $x \geq 0 \leadsto x \to 0^+$ so

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

if $x < 0 \leadsto x \to 0^-$ so

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$\lim_{x \to 0} \frac{|x|}{x} = \text{DNE}$

Question: Plot $f(x) = \frac{|x|}{x}$.

(5) $\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} = \frac{3 - \sqrt{9}}{9 - 9} = \frac{0}{0}$!!!

Method 1 : Use the identity : $a^2 - b^2 = (a-b)(a+b) \leadsto 9 - x = (3 - \sqrt{x})(3 + \sqrt{x})$

so $\lim_{x \to 9} \frac{3 - \sqrt{x}}{(3 - \sqrt{x})(3 + \sqrt{x})} = \lim_{x \to 9} \frac{1}{3 + \sqrt{x}} = \frac{1}{3 + \sqrt{9}} = \frac{1}{3+3} = \frac{1}{6}$

Method 2 : Multiplying by Conjugate

Conjugate of $a - b$ is $a + b$
conjugate of $\sqrt{a} - \sqrt{b}$ is $\sqrt{a} + \sqrt{b}$
" " $a - \sqrt{b}$ is $a + \sqrt{b}$
} keep the order but change + to − or vice versa.

• identity again :
$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$
$(a - \sqrt{b})(a + \sqrt{b}) = a^2 - b$
$(\sqrt{a} - b)(\sqrt{a} + b) = a - b^2$

Multiply both top and bottom by the conjugate of $3 - \sqrt{x}$ and use the identity to simplify :

$$\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} \cdot \frac{3 + \sqrt{x}}{3 + \sqrt{x}} = \lim_{x \to 9} \frac{9 - x}{(9 - x)(3 + \sqrt{x})} = \lim_{x \to 9} \frac{1}{3 + \sqrt{x}} = \frac{1}{3 + \sqrt{9}} = \frac{1}{6}$$

Now let's do a different example:

$$\lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0}$$ $\rightsquigarrow$ what is this in the sence of limits?

$\frac{1}{0}$ is infact $\frac{1}{0^+} = \frac{1}{\text{a small number close to } 0}$ for example:

$$\frac{1}{0.01} = \frac{1}{\frac{1}{100}} = 1 \cdot \frac{100}{1} = 100$$

$$\frac{1}{0.001} = \frac{1}{\frac{1}{1000}} = 1000$$

$\vdots$

$$\frac{1}{0.00\ldots01} = 100\ldots0$$

As $x \to 0^+$, $\frac{1}{0^+}$ keeps getting larger and larger

Math Notation: $+\infty$

so $\boxed{\lim_{x \to 0^+} \frac{1}{x} = \infty}$

and

$$\underbrace{\lim_{x \to 0^-} \frac{1}{x}}$$

Exactly the same process

$$\frac{1}{-0.01} = -100$$

$$\frac{1}{-0.001} = -1000$$

$\vdots$

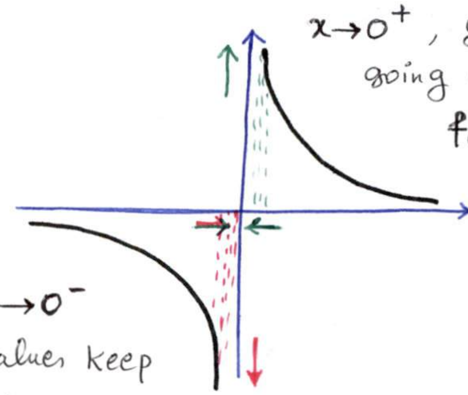
$$\frac{1}{-0.00\ldots01} = -100\ldots0$$

As $x \to 0^-$, $\frac{1}{0^-}$ keeps $\underbrace{\text{getting largely negative}}_{-\infty}$

So $\boxed{\lim_{x \to 0^-} \frac{1}{x} = -\infty}$

(*) $\pm\infty$ is NOT a real number. These are math notations to express very large or very small quantities usually in the context of limits.

Check the above limits graphically. $f(x) = \frac{1}{x}$ looks as follows:



$x \to 0^+$, y-values keep going up so $f(x) \to \infty$

$x \to 0^-$ y-values keep going down so $f(x) \to -\infty$

So: $\rightsquigarrow \lim_{x \to 0} \frac{1}{x} = \text{DNE}$

The limit DNE, because one-sided limits are NOT equal.

Example : What about $\lim\limits_{x \to 0} \frac{1}{x^2} =$

A. ∞ B. $-\infty$ C. DNE D. $\pm\infty$

- $\lim\limits_{x \to 0^+} \frac{1}{x^2} = \frac{1}{(0^+)^2} = \frac{1}{0^+} = \infty$

$\Rightarrow \boxed{A}$

- $\lim\limits_{x \to 0^-} \frac{1}{x^2} = \frac{1}{(0^-)^2} = \frac{1}{0^+} = \infty$

$\frac{1}{(-0.01)^2} = \frac{1}{+}$

⊛ What is the mathematical name for the vertical line $x = 0$ in the two previous examples? Vertical Asymptote

Next Class $\rightsquigarrow$ Asymptotes.

Limit : Practice Problems

(1) $\lim\limits_{x \to 2} \dfrac{x^2+x+6}{x-2}$

(2) $\lim\limits_{x \to 4} \dfrac{x^2-4x}{x^2-3x-4}$

(3) $\lim\limits_{x \to 5} \dfrac{\sqrt{x-1}-2}{x-5}$

(4) $\lim\limits_{x \to 0} \left(\dfrac{1}{5+x} - \dfrac{1}{5-x} \right)$

(5) $\lim\limits_{x \to -4} \dfrac{\sqrt{x^2+9}-5}{x+4}$

(6) $\lim\limits_{x \to 4} \dfrac{(x-4)^3}{|x-4|}$