Last Class

Lecture 12, 13
Oct 1, 3

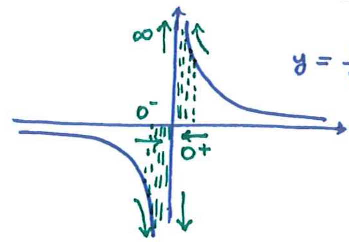
Reminder: Quiz 2
Wednesday, Oct 10
Week 4, 5: limit, V.A, H.A

$$\lim_{x\to 0} \frac{1}{x} = \frac{1}{0} = \pm\infty\ ?$$

$$\lim_{x\to 0^+} \frac{1}{x} = \frac{1}{\text{small + number}} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x} = \frac{1}{\text{small} - \text{number}} = -\infty$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{x} = \text{DNE}$$

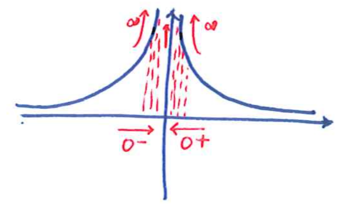


$$\lim_{x\to 0} \frac{1}{x^2} = \frac{1}{(0)^2} = \infty$$

$$\lim_{x\to 0^+} \frac{1}{x^2} = \frac{1}{(\text{small} +)^2} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x^2} = \frac{1}{(\text{small} -)^2} = \infty$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{x^2} = \infty$$

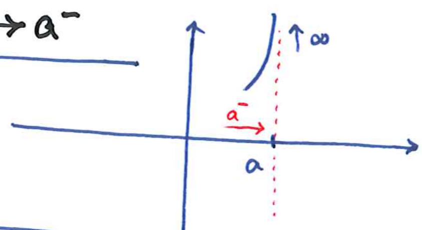


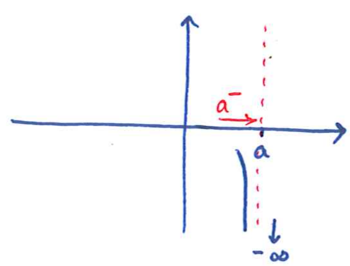
DNE vs. $\pm\infty$

The convention is that DNE is used for the case when left and right limits are NOT equal. Although $\pm\infty$ is NOT a finite number, we consider them as a separate category of limits with value ∞ or $-\infty$.

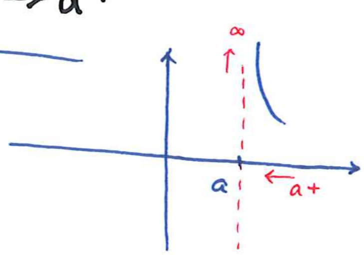
Definition (Vertical Asymptote) We say that a function has a vertical asymptote (VA) at $x=a$ if one of (or both) the following holds:

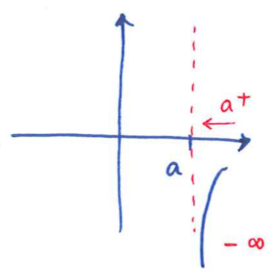
- $\boxed{\lim_{x\to a^-} f(x) = \infty \text{ or } -\infty}$





- $\boxed{\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty}$

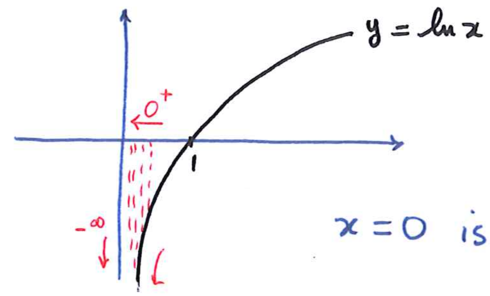




Check above: by definition of V.A.
$f(x) = \frac{1}{x}$ has a V.A. at $x=0$
also
$x=0$ is a V.A. of $f(x) = \frac{1}{x^2}$

Do you know any familiar function with a vertical asymptote? $f(x) = \ln x$: Recall its graph

$\ln x$ is undefined at $x=0$ i.e. $\ln 0 = \infty$, this suggests that $x=0$ can be a V.A. and it is clear from the graph $\Rightarrow$ $\boxed{\lim_{x \to 0^+} \ln x = -\infty}$



$x = 0$ is V.A.

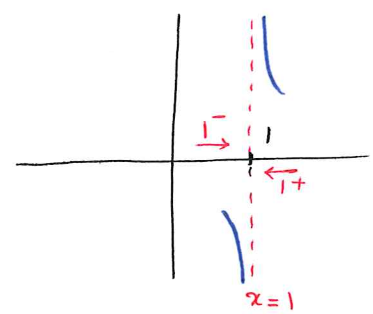
<u>ClickerQ</u> : Which one is the V.A. of $f(x) = \dfrac{x(x+2)}{x-1}$?

A. $x=0$ B. $x=-2$ C. $x=1$ D. Not sure how to solve it.

What x-values make the function undefined? Yes! $x=1$ because the denominator becomes 0. We need to <u>verify</u> by finding the limits at 1.

- $\displaystyle \lim_{x \to 1^+} \frac{x(x+2)}{x-1} \overset{\text{sub}}{=} \frac{1(1+2)}{\underbrace{1^+}_{1.1}-1} = \frac{3}{0^+} = \infty$

- $\displaystyle \lim_{x \to 1^-} \frac{x(x+2)}{x-1} \underset{\text{sub}}{=} \frac{1(1+2)}{\underbrace{1^-}_{0.9}-1} = \frac{3}{0^-} = -\infty$



$\Rightarrow$ $x=1$ is a V.A. of $f(x)$.

<u>Important Note</u> : Candidates for V.A. of $f(x)$ are those x-values that make the function undefined. (those that are NOT in the domain of f). In particular, if $f(x) = \dfrac{P(x)}{q(x)}$ then the candidates are x's such that $q(x) = 0$ and f becomes undefined.

BUT, note these are only candidates, it's possible that f is undefined and still the limit exists and it's NOT $\pm\infty$.

Example. Find all vertical asymptotes of

$$f(x) = \frac{x^2+x}{x^2-x-2}$$

Candidates for V.A. are when $x^2-x-2=0 \Rightarrow (x-2)(x+1)=0$

Verify each:

$\Rightarrow x=2,\ x=-1$

- $\lim\limits_{x\to -1^+} \dfrac{x^2+x}{x^2-x-2} \overset{\text{sub}}{=} \dfrac{(-1)^2+(-1)}{(-1)^2-(-1)-2} = \dfrac{0}{0}$!!! $\rightsquigarrow$ factor

$$= \lim_{x\to -1^+} \frac{x\cancel{(x+1)}}{(x-2)\cancel{(x+1)}} = \frac{-1}{-1-2} = \frac{1}{3} \rightsquigarrow \text{It's NOT } \infty \text{ or } -\infty$$

Similar calculation when $x \to -1^-$

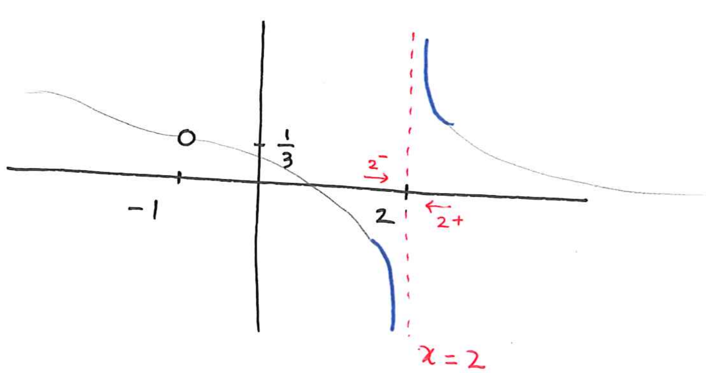
$\Longrightarrow x=-1$ is NOT a V.A. of $f(x)$.

- $\lim\limits_{x\to 2^+} \dfrac{x^2+x}{x^2-x-2} = \dfrac{2^2+2}{2^2-2-2} = \dfrac{6}{0^+} \rightsquigarrow$ This is $+\infty$ $\rightsquigarrow$ V.A.

$$\lim_{x\to 2^-} \frac{x^2+x}{x^2-x-2} = \lim_{x\to 2^-} \frac{x(x+1)}{(x-2)(x+1)} = \frac{6}{\underbrace{(2^- -2)}_{0^-}3} = \frac{6}{0^-} = -\infty$$

$\Longrightarrow x=2$ is a V.A.

How does the function look around 2 and -1?

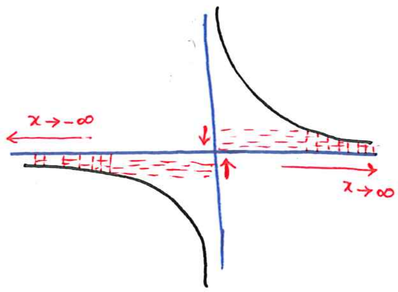


Now let's compute a different limit:

<u>Clicker Q</u>. $\lim_{x \to \infty} \frac{1}{x} =$

A. ∞ B. $-\infty$ C. 0 D. DNE E. NO idea.

Check the graph:



As x gets larger, values of the function get closer to zero

So $\lim_{x \to \infty} \frac{1}{x} = 0$

Check some numbers:

$x \to \infty \xrightarrow{\text{big } x} x = 1000 \rightsquigarrow \frac{1}{x} = \frac{1}{1000}$

$x = 10000 \rightsquigarrow \frac{1}{x} = \frac{1}{10000}$

$\vdots$

$x = 100\dots0 \rightsquigarrow \frac{1}{x} = \frac{1}{100\dots0}$

$\downarrow$ 0

also $x \to -\infty$

$\rightsquigarrow x = -1000 \rightsquigarrow \frac{1}{x} = -\frac{1}{1000}$

$x = -10000 \rightsquigarrow \frac{1}{x} = -\frac{1}{10000}$

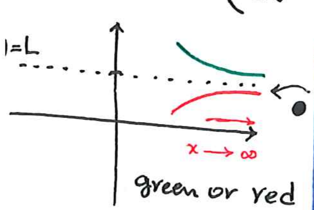
$\vdots$

$x = -100\dots0 \rightsquigarrow \frac{1}{x} = -\frac{1}{100\dots0}$

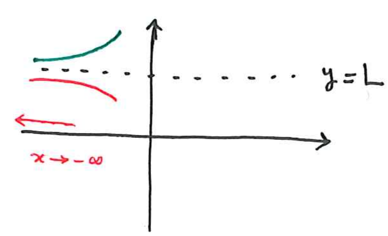
$\downarrow$ 0

- What does it mean for a function to have a horizontal asymptote? It means when x gets larger and larger (also largely negative) then the values of $f(x)$ get closer and closer to some finite number (for example 0 for $\frac{1}{x}$)

Definitio (Horizontal Asymptote) If for a function $f(x)$, one of (or both) the following limits happen



$$\boxed{\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L}$$



then the horizontal line $y = L$ is called the horizontal asymptote (H.A.) of $f(x)$.

Example. Find H.A. of the following functions.

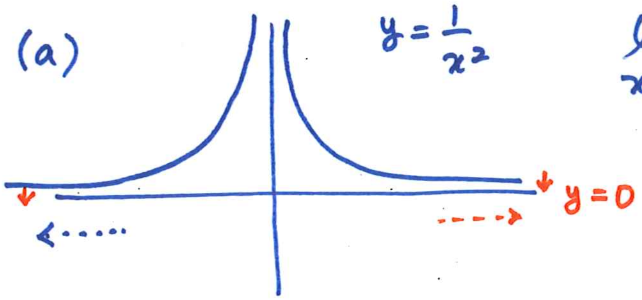
(a) $f(x) = \frac{1}{x^2}$, (b) $f(x) = -x^2+7$

(c) $f(x) = \dfrac{-x^2+7}{2x^2+5x-1}$

(d) $f(x) = \dfrac{-2}{e^x+3}$

(e) $f(x) = -e^{-x}$

(a)



$y = \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{1}{x^2} = 0$

$\lim_{x\to-\infty} \frac{1}{x^2} = 0$

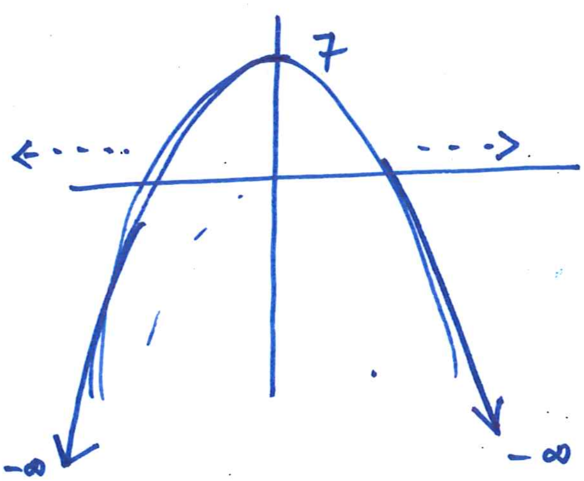
$\Rightarrow$ $y=0$ H.A.

without graph $\lim_{x\to\infty} \frac{1}{x^2} = \dfrac{1}{\text{very large number}} \Rightarrow 0 \Rightarrow y=0$ H.A.

(b) $\lim_{x\to\infty} -x^2+7 = \lim_{x\to\infty} -x^2 = -(\text{large number})^2$

$-x^2$: dominant term when $x\to\infty$

$= -\infty$ → NOT finite L



NO H.A.

$\lim_{x\to-\infty} -x^2+7 = -(\cancel{-\infty}^{\,\infty})^2 = -\infty$

(*) Polynomials do NOT have H.A.
because as $x\to\pm\infty$ $f(x)\to\pm\infty$.

(c) $$\lim_{x \to \infty} \frac{-x^2 + 7}{2x^2 + 5x - 1} = \lim_{x \to \infty} \frac{-\cancel{x^2}\left(1 + \frac{7}{-x^2}\right)}{2\cancel{x^2}\left(1 + \frac{5\cancel{x}}{2x^{\cancel{2}}} - \frac{1}{2x^2}\right)}$$

dominant terms

Factor dominant terms

Recall:

$\lim_{x \to \infty} \frac{7}{x^2} = 0$

$\lim_{x \to \infty} \frac{5}{x} = 0$

$\lim_{x \to \infty} \frac{1}{2x^2} = 0$

$$= \lim_{x \to \infty} \frac{-\left(1 - \frac{7}{x^2}\right)}{2\left(1 + \frac{5}{2x} - \frac{1}{2x^2}\right)} = \frac{-1}{2}$$

0 0 0

$\Rightarrow y = -\frac{1}{2}$ is a H.A. of f

Similar steps when $x \to -\infty$ (Same limit)

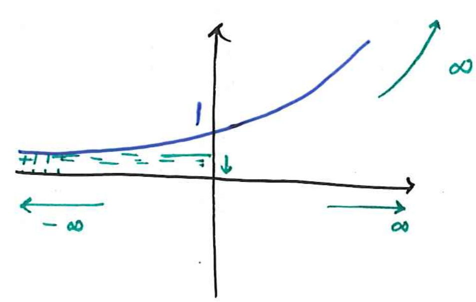
* For finding the limits of polynomials at $\pm\infty$, find the dominant term, and cancel them if there is any common factors and check how the remaining terms behave as $x \to \pm\infty$.

(d_1) Simpler case:

$$\boxed{\lim_{x \to \infty} e^x = \infty \qquad \lim_{x \to -\infty} e^x = 0}$$

→ very important

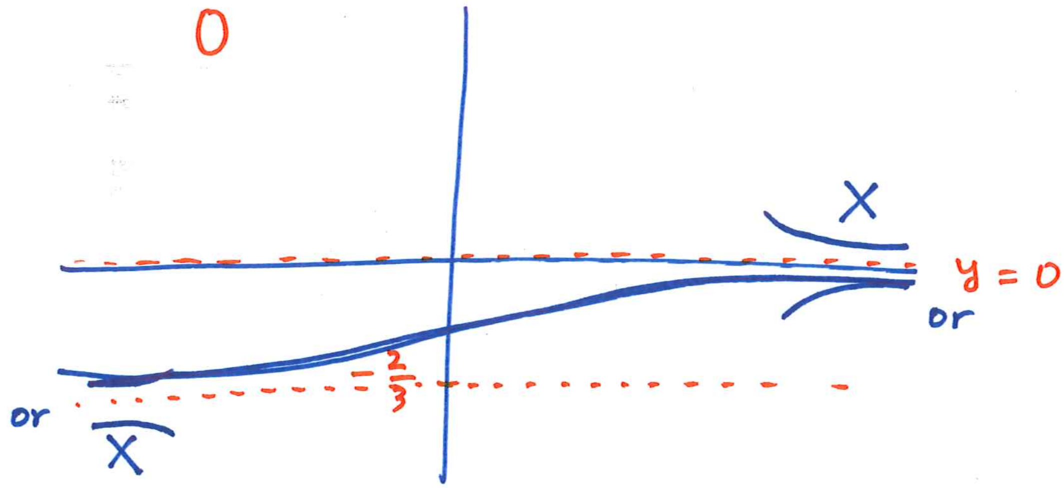
Check the graph



still $y = 0$ is a H.A. of $y = e^x$ on the left.

(d) $$\lim_{x\to+\infty} \frac{-2}{\underbrace{e^x}_{\infty}+3} = \frac{-2}{\infty+3} = \frac{-2}{\infty} \to 0$$

$$\lim_{x\to-\infty} \frac{-2}{\underbrace{e^x}_{0}+3} = \frac{-2}{0+3} = -\frac{2}{3}$$



(e) $$\lim_{x\to\infty} -e^{-x} = -\underbrace{e^{-\infty}}_{0} = 0$$

$$\lim_{x\to-\infty} -e^{-x} = -e^{-(-\infty)} = -\underbrace{e^{\infty}}_{\infty} = -\infty$$