Reminder :

- Quiz 2 : Wednesday , Oct 10 ,
  - → Week 4 - 5 : limits and asymptotes (today's lecture not included
  - → Check Canvas for the posted lectures .
  - → One computational question such as finding limits & asymptote
    - ↳ Check posted practice problems from the text book
  - → One graphical interpretation Question
    - ↳ Lab's worksheet and examples done in class .

Summary of What we learned about limits and asymptotes :

(1) Meaning of $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = DNE$

* As x approaches a from right and left $f(x)$ approaches L

when $x \to a^+$ and $x \to a^-$ we get two different values for the limit

(2) Difference between $\lim_{x \to a} f(x)$ and $f(a)$

This is the value of the function close to a .

↳ This is the exact value of function when $x = a$

(3) First step in finding a limit ,

↳ Direct substitution

(4) What if we get $\frac{0}{0}$? ↳ Factor, simplify , multiply by conjugate ... to make the problem appear and then cancel it

(5) $\lim_{x \to a} f(x) = \pm \infty$ and Vertical Asymptote

→ computational function undefined

→ graphical

↳ vertical tails around $x = a$ upward / downward

(6) $\lim_{x \to \pm\infty} f(x) = L$ and Horizontal Asymptote .

→ Computational dominant term

→ graphical

↳ Horizontal tails around $y = L$

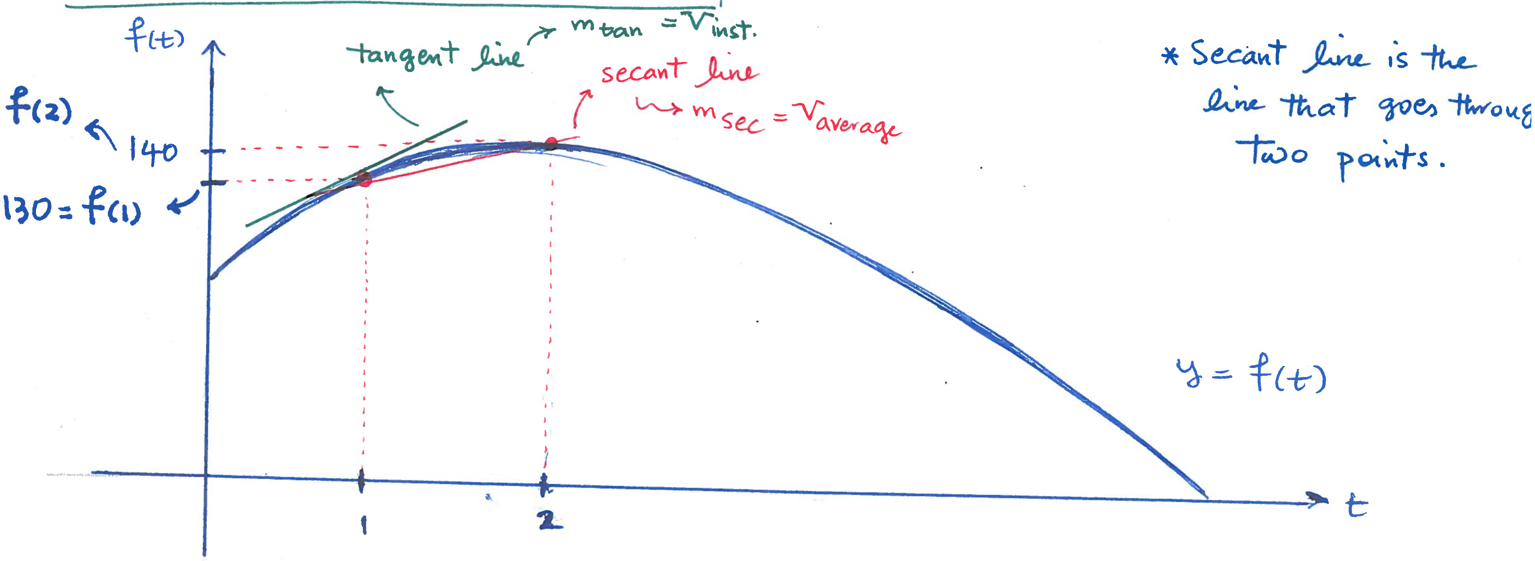
→ Check the learning goals posted for the topic of limits and asymptotes and make sure you have understood the concepts to solve the problems associated to each goal .

Today : How to find slope of the tangent line to a function $y = f(x)$.

Example : Suppose you are on top of a 100 ft building and you throw a ball upward. By law of physics, the motion of this ball can be described by a quadratic function. Take position (height) of the ball : f, f is a function of time : t so we have $f(t)$.

Now let's assume height is expressed by the function:

$\boxed{f(t) = -10t^2 + 40t + 100}$ → a parabola : vertex: $(2, 140)$



Q1 : What is the average velocity between $t = 1$ and $t = 2$?

Average velocity : $V_{ave} = \dfrac{\text{Change in height}}{\text{Change in time}} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{140 - 130}{1} = 10$ ft/sec

$= \dfrac{\Delta f}{\Delta h} = \dfrac{\text{Rise}}{\text{Run}}$ = the slope of the line through $(1, 130)$ and $(2, 140)$

= the slope of the secant line

Q2 : What is the instantaneous velocity at $t = 1$?

This is NOT as easy as V_{ave}, since we have only one point and we cannot find Δf and Δt. Instantaneous velocity is the slope of the line that passes through (1, 130) and touches the graph only at the point (1, 130). This line is called the tangent line to the graph of f(x) at the given point. We need to find the slope of the tangent line (m_{tan}) and to do that we use m_{secant} as approximations for m_{tan} and we keep making our approx. better by getting closer and closer to (1, 130)

time interval	Δt	Δf	$V_{ave} = \frac{\Delta f}{\Delta t} = m_{sec}$
[1, 2]	$2 - 1 = 1$	$f(2) - f(1) = 140 - 130 = 10$	$\frac{10}{1} = 10 \approx V_{inst}$
[1, 1.5]	$1.5 - 1 = 0.5$	$f(1.5) - f(1) = 137.5 - 130 = 7.5$	$\frac{7.5}{0.5} = 15 \approx V_{inst}$
[1, 1.15]	$1 - 1.15 = 0.15$	$f(1.15) - f(1) = 132.775 - 130 = 2.775$	$\frac{2.775}{0.15} = 18.5 \approx V_{inst}$
[1, 1.01]	$1 - 1.01 = 0.01$	$f(1.01) - f(1) = 130.199 - 130 = 0.199$	$\frac{0.199}{0.01} = 19.9 \approx V_{inst}$
[1, 1.001]	$1 - 1.001 = 0.001$	$f(1.001) - f(1) = 0.01999$	$\frac{0.01999}{0.001} = 19.99 \approx V_{inst}$

Conclusion: As Δt shrinks, $m_{sec} = \frac{\Delta f}{\Delta t}$ gets closer and closer to 20

so $V_{inst} = 20$

Summary of what we did

$V_{ave} = \frac{\Delta f}{\Delta t}$ = slope of the secant line through two points

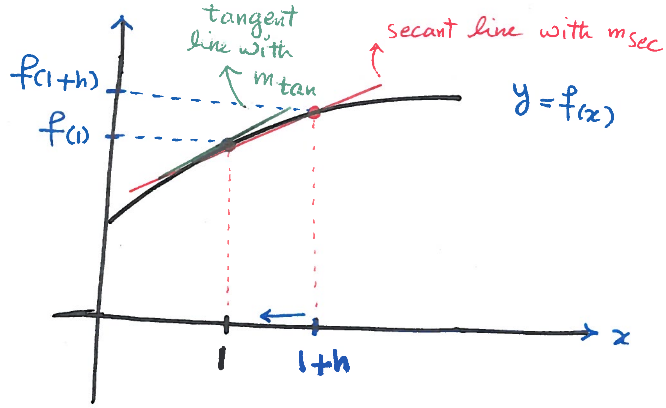
V_{inst} = slope of the tangent line at one point $\xrightarrow{\text{How to}}$ Compute ?

Shrink the time interval

↪ two points become one

↪ secant line becomes tangent

Let's formulate this process of Shrinking mathematically :



Goal : Find the slope of the tangent line : m_{tan} at $x=1$ to some graph $y=f(x)$

- Pick an arbitrary point close to 1 say $1+h$ where h is a smaller number ($h=0.5, 0.15, 0.01, \dots$ in last example)
- Find the slope of the secant line through the two points $(1, f(1))$ and $(1+h, f(1+h))$

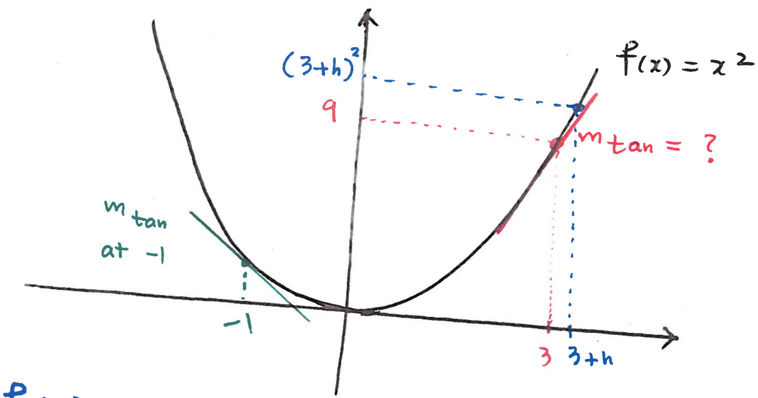
$$m_{sec} = \frac{\text{Rise}}{\text{Run}} = \frac{f(1+h)-f(1)}{1+h-1} = \frac{f(1+h)-f(1)}{h}$$

- Shrink the interval on the x-axis by making h smaller & smaller : $h \longrightarrow 0$
- The point $(1+h, f(1+h))$ gets closer and closer to $(1, f(1))$ so secant line almost becomes the tangent line, this means m_{sec} gets closer and closer to be m_{tan} :

As $h \to 0$; $m_{sec} \longrightarrow m_{tan}$

translate :
$$\boxed{\lim_{h\to 0} m_{sec} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = m_{tan} \text{ at } x=1}$$

Example 2. Find slope of the tangent line to $f(x) = x^2$ at the point $(3, 9)$



m_{tan} at $x=3$: $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{3+h-3}$

$f(x) = x^2$

$$= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$

substitute

$$= \frac{(3+0)^2 - 3^2}{0} = \frac{0}{0}$$

algebra

expand the square

$$= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h}$$

factor h

$$= \lim_{h\to 0} \frac{h(6+h)}{h}$$

cancel

re-substitute

$$= \lim_{h\to 0} 6 + h = 6 + 0 = 6$$

So the slope of the tangent line at $x=3$ is 6.

We can find the above limit to find m_{tan} for any other point say "a" we just need to replace 3 by any number "a"

m_{tan} at $x=a$: $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{a+h-a} = \lim\limits_{h\to 0} \dfrac{(a+h)^2 - a^2}{h}$

$f(x) = x^2$