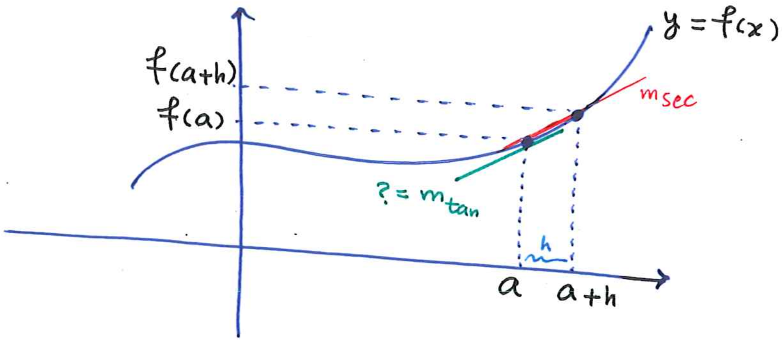
Lecture 15
Oct 10

Last Class : How to find the slope of the tangent line to the graph of a function $y = f(x)$ at some given point.



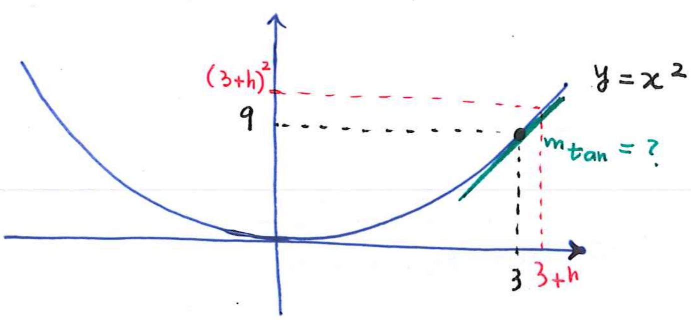
$$m_{sec} = \frac{\text{Rise}}{\text{Run}} = \frac{f(a+h) - f(a)}{a+h-a}$$

Shrink h $\rightsquigarrow$ secant line becomes almost the tangent line so $m_{sec} \longrightarrow m_{tan}$

$$\Rightarrow \quad m_{tan} = \lim_{h \to 0} m_{sec} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

We solved

Example : Slope of the tangent line to $f(x) = x^2$ at $x = 3$?



$$\underset{\text{at } x=3}{m_{tan}} = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

so if $f(x) = x^2$ we have

$$\lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \frac{(3+0)^2 - 3^2}{0} = \frac{0}{0} \ !!!$$

expand the square

$$\lim_{h\to 0} \frac{3^2 + 6h + h^2 - 9}{h} = \lim_{h\to 0} \frac{h(6+h)}{h}$$

$$= \lim_{h\to 0} 6 + h = 6$$

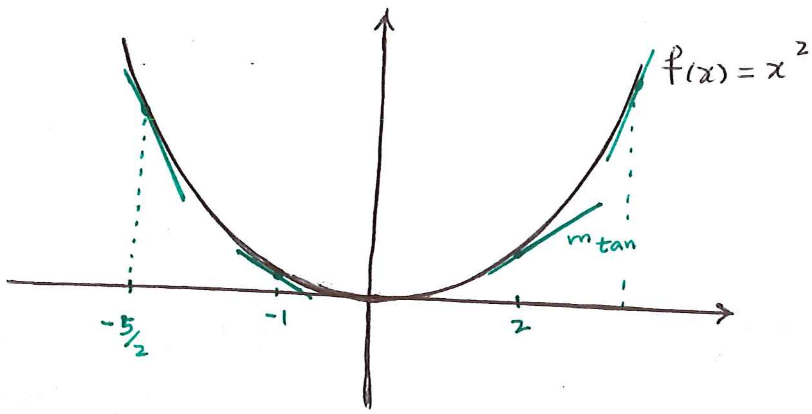
$$\boxed{m_{tan} = 6}$$

Similarly, if we are asked to find slope of the tangent line at

$x = 2 \rightsquigarrow \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

at $x = -1 \rightsquigarrow \lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h}$

at $x = -\frac{5}{2} \rightsquigarrow \lim_{h \to 0} \dfrac{f(-5/2+h) - f(-5/2)}{h}$



* At any x, we can find the slope of the tangent line by using the formula and finding the limit.

This suggests that we can look at the process of finding the slope of the tangent line to $f(x) = x^2$ (or any other function) as a function machinery.

input variable x ⟶ output $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

With this function we can find m_{tan} for $f(x) = x^2$ in one go for all x-values:

$f(x) = x^2$

→ Slope of the tangent line to $f(x)$ at any x $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = x^2$

$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$ sub 0 for h $= \frac{(x+0)^2 - x^2}{0} = \frac{0}{0}$!

expand the square

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

cancel x^2

$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \frac{0+0}{0}$!

Factor h

$= \lim_{h \to 0} \frac{h(2x+h)}{h}$ still $\frac{0}{0}$

cancel h

$= \lim_{h \to 0} 2x + h$

re-sub 0 for h

$= 2x + 0 = 2x$

At any point x ; the slope of the tangent line to $f(x) = x^2$ is $2x$.

So slope of tangent line at $x = 2 \rightsquigarrow 2 \times 2 = 4$

m_{tan} at $x = -1 \rightsquigarrow 2 \times (-1) = -2$

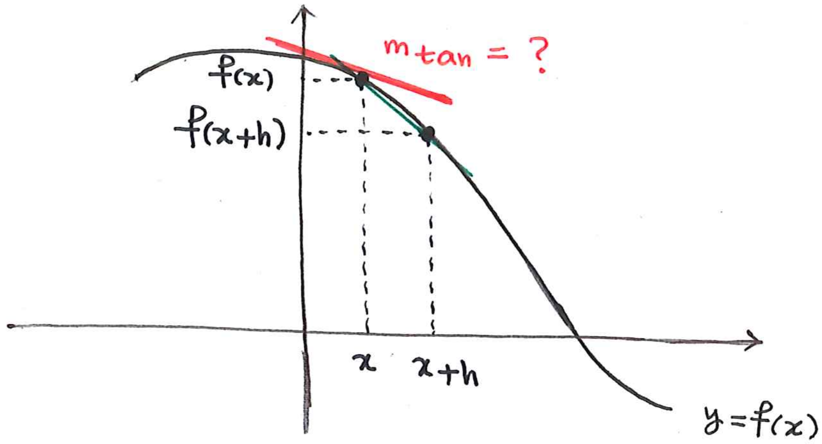
" " $x = -5/2 \rightsquigarrow 2 \times (-5/2) = -5$

Derivative is just the Slope of the tangent line so

If $f(x) = x^2$ then derivative of $f(x) = 2x$

- derivative at $x = 2$ is 4
- derivative at $x = -5/2$ is -5

In general, for any function $f(x)$:



- What is derivative? It's the slope of the tangent line to a function $f(x)$

- Mathematical Definition of the derivative

$$\boxed{f'(x) = m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$ → The limit definition of the derivative

- Notation: $y = f(x)$ is the function

its derivative is $f'(x)$, $\frac{df}{dx}$, y', $\frac{dy}{dx}$

These notations clarify that f or y are functions of x.

- The quotient $\frac{f(x+h) - f(x)}{h}$ is called difference quotient.

so if $f(x) = x^2$ we computed that $f'(x) = 2x$

therefore; $f'(2) = 4$

$f'(-1) = -2$

$f'(-5/2) = -5$

$\vdots$

So far we found that:

$$f(x) = x^2 \rightsquigarrow f'(x) = 2x$$

Now let's compute:

<u>Example</u>: Find the derivative of $f(x) = x^3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x)=x^3$

sub

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \frac{(x+0)^3 - x^3}{0} = \frac{0}{0}$$

expand the cubic

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$(x+h)^3 = (x+h)^2(x+h)$$
$$= (x^2+2xh+h^2)(x+h$$

cancel x^3

still not working!

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \frac{0}{0}$$

factor h

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

cancel h

re-sub 0 for h

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2 + 0 + 0 = \boxed{3x^2}$$

So $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$\Rightarrow f'(2) = 3 \cdot (2)^2 = 12 \rightsquigarrow m_{tan}$ at 2

$f'(-1) = 3 \cdot (-1)^2 = 3 \rightsquigarrow m_{tan}$ at -1