- HW 2 is in MLC for return ⇝ grades on Canvas
- Quiz 2 and solution posted on Canvas.

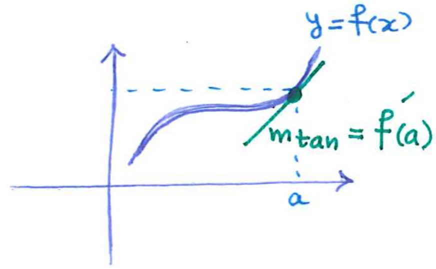
__Last class__ : $y = f(x)$ a given function

→ Derivative of $f(x)$: the slope of the tangent line to the graph of $f(x)$ at any x

→ Math definition of the derivative :

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

→ Notation : $\frac{df}{dx}$ or y'



Examples :

(1) $f(x) = x^2 \rightsquigarrow f'(x) = ? = 2x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{0}{0} ! \quad \text{expand}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h} = 2x$$

(2) $f(x) = x^3 \rightsquigarrow f'(x) = ? = 3x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \frac{0}{0}$$

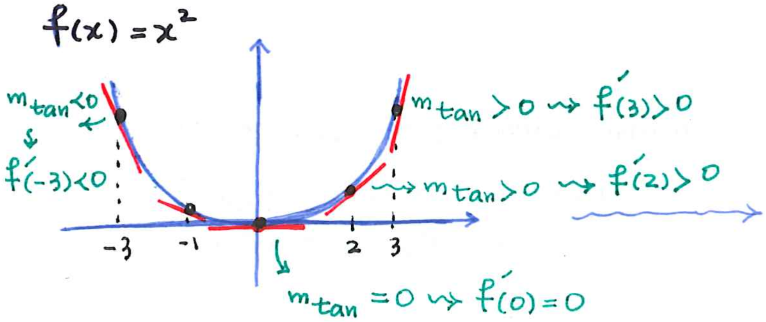
expand

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

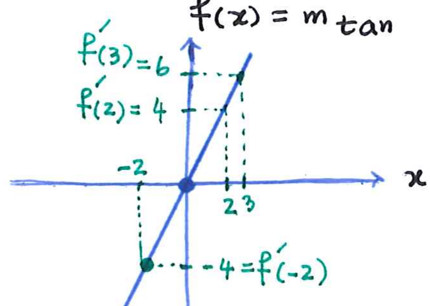
$$= \cdots \text{ algebra}$$

$$= 3x^2$$

$f(x) = x^2$



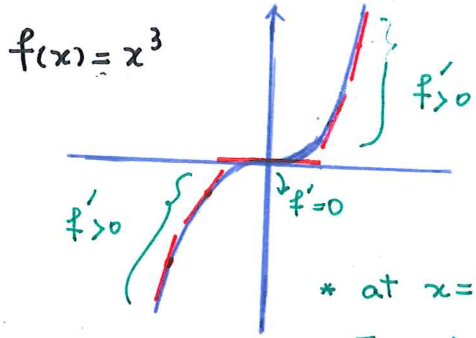
$f'(x) = m_{tan}$



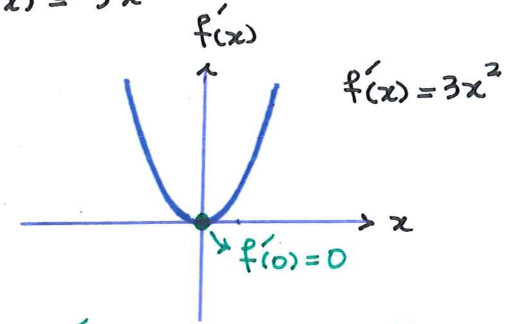
* For each x value find m_{tan} at that x value and make a graph for m_{tan} or for $f'(x)$

$f'(2) = 2 \cdot 2 = 4$, $f'(3) = 2 \cdot 3 = 6$, $f'(-2) = 2 \cdot (-2)$

Graph of $f(x) = x^3$ and its derivative $f'(x) = 3x^2$

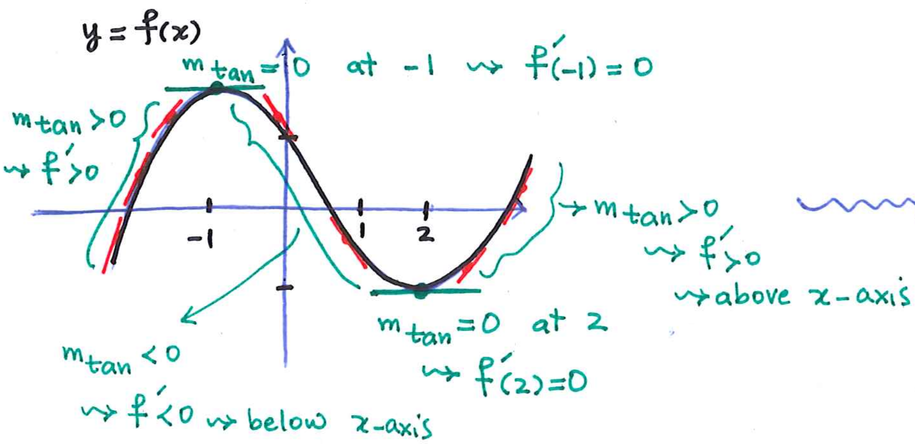


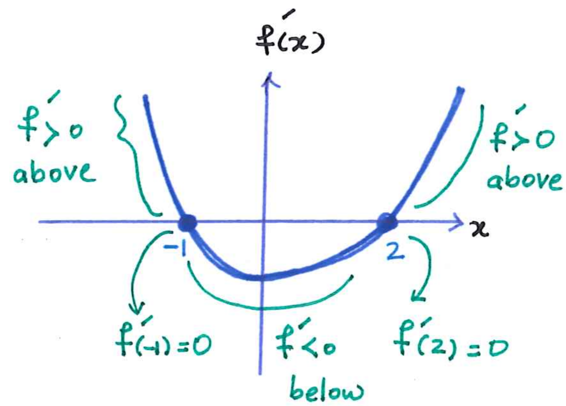
track m_{tan} and graph m_{tan} vs. x



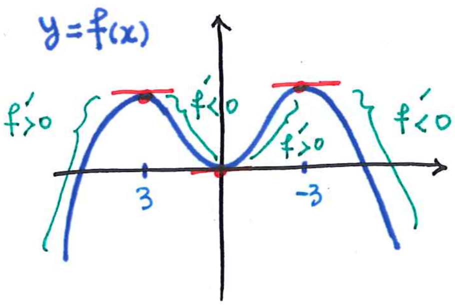
* at $x = 0 \rightsquigarrow m_{tan} = 0 = f'(0) \rightsquigarrow$ graph of f' crosses x-axis at 0
* Everywhere else $m_{tan} > 0$ so $f'(x) > 0 \rightsquigarrow$ graph of f' above x-axis.

Example 1: Using the fact that $f'(x)$ is the slope of the tangent line, sketch the graph of $f'(x)$ for the given function $f(x)$.

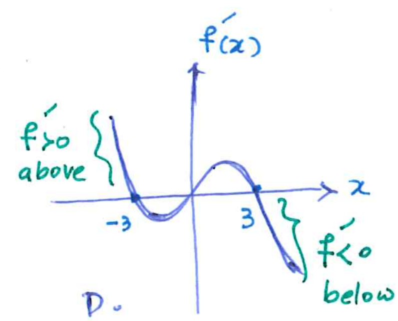




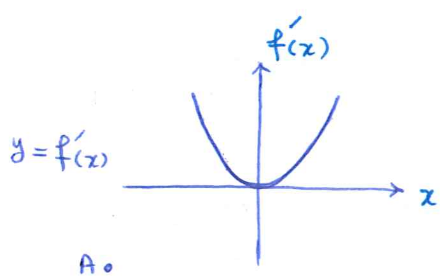
Clicker Q. Which one is the graph of $f'(x)$ for the following f.



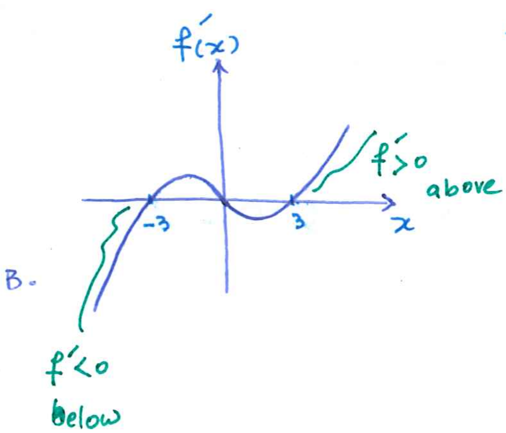
* $m_{tan} = 0$ at 3 points
  $\Rightarrow f'$ crosses x-axis at 3 points
  $\Rightarrow$ B or D
* In the interval $(-\infty, 3) \Rightarrow m_{tan} > 0$
  $\Rightarrow f' > 0 \Rightarrow$ above
  $\Rightarrow \boxed{\text{only D.}}$



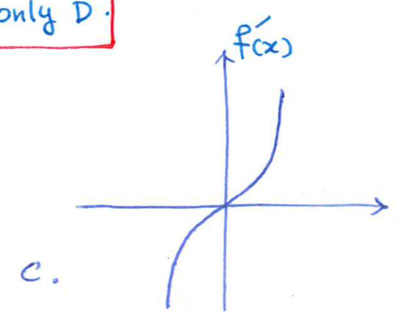
D.



A.



B.



C.

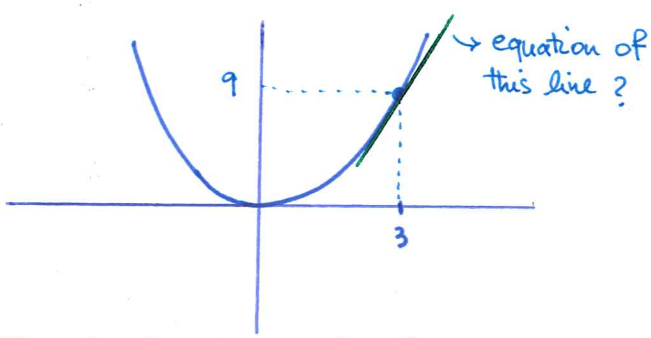
Conclusion → Relationship between f and f' graphically :

If f is increasing $\Leftrightarrow m_{tan} > 0 \Leftrightarrow f'$ positive
above x-axis

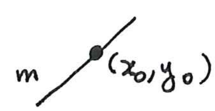
If f is decreasing $\Leftrightarrow m_{tan} < 0 \Leftrightarrow f'$ negative
below x-axis

If f is constant $\Leftrightarrow f$ has a horizontal tangent line $\Leftrightarrow m_{tan} = f' = 0$

Example 2. Find the equation of the tangent line to $f(x) = x^2$ at $x = 3$.



Recall : For the equation of a line we need
slope m — a point on the line (x_0, y_0)



$\Rightarrow \quad y - y_0 = m(x - x_0)$

* This line is the tangent line so

$m = m_{tan}$ at $3 = f'(3)$

point ⇝ tangency point : $(3, f(3)) = (3, 9)$

We already know if $f(x) = x^2$ then $f'(x) = 2x$ so

m_{tan} at $3 = f'(3) = 2 \cdot 3 = 6$

$\left.\begin{array}{l} m_{tan} = 6 \\ \text{point} : (\underset{x_0}{3}, \underset{y_0}{9}) \end{array}\right\} \Rightarrow y - y_0 = m(x - x_0) \Rightarrow y - 9 = 6(x - 3)$

$\Rightarrow \boxed{y = 6x - 9}$

Example 3: Find the equation of the tangent line to the graph of the function $f(x) = \dfrac{x}{x-3}$ at $x = 4$.

We need m_{tan} at 4 and a point

$\hookrightarrow m_{tan} = f'(x) \xrightarrow{x=4} m_{tan} = f'(4)$

$\hookrightarrow$ tangency point : $(4, f(4))$

$$f(4) = \frac{4}{4-3} = \frac{4}{1} = 4 \Rightarrow \text{point: } (4,4)$$

How to find $f'(4)$? For now, use the limit definition of the derivative.

$$f(x) = \frac{x}{x-3} \Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\xrightarrow{x=4} f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$$

plug in 4+h and 4 into f(x)

$$= \lim_{h\to 0} \frac{\frac{4+h}{4+h-3} - \frac{4}{4-3}}{h}$$

Simplify

$$= \lim_{h\to 0} \frac{\frac{4+h}{1+h} - 4}{h}$$

take the Common denominator

Be careful about the – in between

$$= \lim_{h\to 0} \frac{\frac{4+h-4(1+h)}{1+h}}{h}$$

Simplify

$$= \lim_{h\to 0} \frac{4+h-4-4h}{1+h} \cdot \frac{1}{h}$$

Cancel & re-sub

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{1+h} \cdot \frac{1}{\cancel{h}} = -3$$

So we found

$$f'(4) = -3 = m_{tan}$$

tangency point $(4, 4)$

$$y - y_0 = m(x - x_0)$$

$$\Rightarrow y - 4 = -3(x-4) \Rightarrow \boxed{y = -3x + 16}$$