Last Class : Derivative Rules.

- Recall : Derivative of $y = f(x)$: $f'(x)$ is the slope of the tangent line to the graph of $f(x)$ at the point $(x, f(x))$.

→ Long way of finding $f'(x)$: limit definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

→ Short way of finding $f'(x)$: Derivative Rules

Family of functions

- power functions

$$(x^n)' = nx^{n-1} \longrightarrow \text{Power Rule}$$

- exponential functions

$$(b^x)' = b^x \cdot \ln b$$

$$\Rightarrow (e^x)' = e^x$$

- trig functions

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$
$$= 1 + \tan^2 x$$

Operations between functions

- Sum, difference

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

- Constant multiple

$$(c\, f(x))' = c\, f'(x)$$

a constant number

but $(c)' = 0$

Clicker Q: $f(x) = 5x^2 + \sqrt[3]{x} - \cos x + 9e^x + \frac{2}{x} - 5^x - 1$, $f'(x) = ?$

A. $10x + \frac{1}{3\sqrt[3]{x}} - \sin x + 9e^x - \frac{2}{x^2} - 5^x - 1$

B. $2x + \frac{1}{3\sqrt[3]{x^2}} + \sin x + 9e^x + \frac{2}{x^2} - 5^x \cdot \ln 5 - 1$

C. $10x + \frac{1}{3\sqrt[3]{x^2}} + \sin x + 9e^x - \frac{2}{x^2} - 5^x \cdot \ln 5$

D. $10x - \frac{1}{3\sqrt[3]{x^2}} - \sin x - 9e^x + \frac{2}{x^3} + 5^x$

Let's go term by term:

- $y = 5x^2 \xrightarrow[\text{multiple}]{\text{Constant}} y' = 5 \cdot 2x = 10x$
- $y = \sqrt[3]{x} = x^{\frac{1}{3}} \xrightarrow[\text{rule}]{\text{power}} y' = \frac{1}{3} x^{\frac{1}{3}-1 = -\frac{2}{3}} = \frac{1}{3x^{2/3}}$
- $y = \cos x \xrightarrow{\text{trig}} y' = -\sin x$
- $y = 9e^x \rightarrow y' = 9e^x$
- $y = \frac{2}{x} = 2x^{-1} \rightarrow y' = 2 \cdot (-1) x^{-1-1} = \frac{-2}{x^2}$
- $y = 5^x \rightarrow y' = 5^x \cdot \ln 5$
- $y = 1 \rightarrow y' = 0$

$\Rightarrow \left(5x^2 + \sqrt[3]{x} - \cos x + 9e^x + \frac{2}{x} - 5^x - 1\right)'$

$= 10x + \frac{1}{3\sqrt[3]{x^2}} + \sin x + 9e^x - \frac{2}{x^2} - 5^x \cdot \ln 5$

$\boxed{C}$

* Don't forget to multiply − sign in the function expression with the − sign comes from the derivative

Clicker Q: $f(x) = \underbrace{x}_{f' \to 1} \cdot \underbrace{\sin x}_{g' \to \cos x} \Rightarrow f'(x) = 1 \cdot \cos x = \cos x$

A. True

$\boxed{\text{B.}}$ False

C. Do NOT know the derivative rule for the product of two functions

Next function operation: $y = f(x) \cdot g(x)$ → two function multiplied

$$\boxed{\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)} \rightarrow \text{Product Rule}$$

We can verify this in an easy example:

$$y = \underbrace{(x+1)}_{f}\underbrace{(2x+1)}_{g}$$

1^{st} method: multiply the two terms and use the power rule:

$$y = 2x^2 + x + 2x + 1 = 2x^2 + 3x + 1 \Rightarrow y' = 4x + 3$$

2^{nd} method: Use the product rule

$$\left.\begin{array}{ll} f(x) = x+1 & g(x) = 2x+1 \\ f'(x) = 1 & g'(x) = 2 \end{array}\right) \Rightarrow y' = f'g + fg'$$

$$= 1(2x+1) + (x+1)\cdot 2$$

$$= 2x + 1 + 2x + 2 = 4x + 3 \checkmark$$

(*) 1^{st} method is NOT always possible

Examples:

(a) $y = x \sin x$

(b) $y = x^2 e^x$

(c) $y = 3\sqrt[3]{x^2} + 7 \sin x \cos x$

(a) $y = \underbrace{x}_{f} \underbrace{\sin x}_{g}$

$f(x) = x$ $\qquad g(x) = \sin x$

$f'(x) = 1$ $\qquad g'(x) = \cos x$

$$\Rightarrow y' = f'g + fg' = 1 \cdot \sin x + x \cdot \cos x = \sin x + x \cos x$$

(b) $y = \underbrace{x^2}_{f}\, \underbrace{e^x}_{g}$

$f(x) = x^2$ $\quad$ $g(x) = e^x$

$f'(x) = 2x$ $\quad$ $g'(x) = e^x$

$\Rightarrow y' = f'g + fg'$

$= 2x \cdot e^x + x^2 \cdot e^x$

factor

$= e^x(2x + x^2)$

(c) $y = \underbrace{3\sqrt[3]{x^2}}_{\text{power function}} + \underbrace{7 \sin x \cos x}_{\text{product rule}}$

$\left(3\sqrt[3]{x^2}\right)' = \left(3x^{\frac{2}{3}}\right)' = 3 \cdot \frac{2}{3} x^{\frac{2}{3}-1} = 2x^{-\frac{1}{3}} = 2 \cdot \frac{1}{x^{\frac{1}{3}}} = \frac{2}{\sqrt[3]{x}}$

$\left(7 \sin x \cos x\right)' = 7\left(\underbrace{(\sin x)'}_{f'}\underbrace{\cos x}_{g} + \underbrace{\sin x}_{f}\underbrace{(\cos x)'}_{g'}\right)$

Constant multiple stays for the whole derivative

$= 7\left(\cos x \cdot \cos x + \sin x \cdot (-\sin x)\right)$

$= 7\left(\cos^2 x - \sin^2 x\right)$

So $y' = \frac{2}{\sqrt[3]{x}} + 7\left(\cos^2 x - \sin^2 x\right)$

Now if I ask: slope of the tangent line to y at $x = \pi$

we evaluate $y'(\pi)$.

$y'(\pi) = \frac{2}{\sqrt[3]{\pi}} + 7\left(\underset{(-1)^2}{\cos^2 \pi} - \underset{(0)^2}{\sin^2 \pi}\right) = \frac{2}{\sqrt[3]{\pi}} + 7$

Next function operation : $y = \dfrac{f(x)}{g(x)}$ → two functions divided

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$$ → Quotient Rule

Example : (HW 3 Question) → with limit definition $y = \dfrac{\overbrace{3x+1}^{f}}{\underbrace{x-1}_{g}}$; find y'

Now with the quotient rule :

$f(x) = 3x+1$ $g(x) = x-1$

$f'(x) = 3$ $g'(x) = 1$

$$y' = \frac{f'\cdot g - f\cdot g'}{g^2} = \frac{3\cdot(x-1) - (3x+1)\cdot 1}{(x-1)^2} = \frac{\cancel{3x} - 3 - \cancel{3x} - 1}{(x-1)^2}$$

$$= \frac{-4}{(x-1)^2}$$

You got this by using the long way.

Example . Prove that $(\tan x)' = \sec^2 x$ by using the Quotient rule.

$$y' = \tan x = \frac{\sin x}{\cos x}$$

$$y' = \frac{(\sin x)' \cos x - \sin x\cdot(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

*Recall :
- Important identity: $\cos^2 x + \sin^2 x = 1$
- also $\dfrac{1}{\cos x} = \sec x$

Your turn :

$$h(x) = \frac{\overset{f}{x e^x}}{\underset{g}{3x+1}}, \text{ find } h'(x)$$

$f(x) = xe^x$

$g(x) = 3x+1$

product rule

$f'(x) = (x)'e^x + x(e^x)'$

$= 1 \cdot e^x + xe^x$

$= e^x + xe^x$

$g'(x) = 3$

$$h'(x) = \frac{f' \cdot g - f \cdot g'}{g^2} = \frac{(e^x + xe^x)\cdot(3x+1) - xe^x \cdot 3}{(3x+1)^2}$$

$$= \frac{\cancel{3xe^x} + e^x + 3x^2e^x + xe^x - \cancel{3xe^x}}{(3x+1)^2}$$

cancel & factor e^x

$$= \frac{e^x(1 + 3x^2 + x)}{(3x+1)^2}$$

* Derivative rules can be combined together in one function. Take a close look and identify the rules and go step by step. Always double check your algebra.

* Quotient Rule is usually followed by simplification steps as we need to distribute the multiplication and cancel terms.

The indicated functions are good practices for Quiz 3 next week.

Derivative Worksheet

1. $f(x) = 4x^5 - 5x^4$

2. $f(x) = e^x \sin x$

3. $f(x) = (x^4 + 3x)^{-1}$

4. $f(x) = 3x^2(x^3+1)^7$

5. $f(x) = \cos^4 x - 2x^2$

6. $f(x) = \dfrac{x}{1+x^2}$

7. $f(x) = \dfrac{x^2-1}{x}$

8. $f(x) = (3x^2)(x^{\frac{1}{2}})$

9. $f(x) = \ln(xe^{7x})$

10. $f(x) = \dfrac{2x^4+3x^2-1}{x^2}$

11. $f(x) = (x^3)\sqrt[5]{2-x}$

12. $f(x) = 2x - \dfrac{4}{\sqrt{x}}$

13. $f(x) = \dfrac{4(3x-1)^2}{x^2+7^x}$

14. $f(x) = \sqrt{x^2+8}$

15. $f(x) = \dfrac{x}{\sqrt{1-(\ln x)^2}}$

16. $f(x) = \dfrac{6}{(3x^2-\pi)^4}$

17. $f(x) = \dfrac{(3x^2-\pi x)^4}{6}$

18. $f(x) = \dfrac{x}{(x^2+\sqrt{3x})^5}$

19. $f(x) = (xe^x)^\pi$

20. ~~$f(x) = \left[\arctan(2x)\right]^{10}$~~

21. $f(x) = (e^{2x}+e)^{\frac{1}{2}}$

22. $f(x) = (x^6+1)^5(4x+7)^3$

23. $f(x) = (7x+\sqrt{x^2+3})^6$

24. $f(x) = \dfrac{\frac{1}{x}+\frac{1}{x^2}}{x-1}$

25. $f(x) = \sqrt[3]{x^2} - \dfrac{1}{\sqrt{x^3}}$

26. $f(x) = \sqrt{\dfrac{2x+5}{7x-9}}$

27. $f(x) = \dfrac{\sin x}{\cos x}$

28. $f(x) = e^x(x^2+3)(x^3+4)$

29. $f(x) = \dfrac{5x^2-7x}{x^2+2}$

30. $f(x) = \left[\ln(5x^2+9)\right]^3$

31. $f(x) = \ln(5x^2+9)^3$

32. $f(x) = \cot(6x)$

33. ~~$f(x) = \sec^2 x \cdot \tan x$~~

34. ~~$f(x) = \arcsin(2^x)$~~

35. $f(x) = \tan(\cos x)$

36. $f(x) = [(x^2-1)^5 - x]^3$

37. ~~$f(x) = \sec x \cdot \sin(3x)$~~

38. $f(x) = \dfrac{(x-1)^3}{x(x+3)^4}$

39. $f(x) = \log_5(3x^2+4x)$

If f and g are differentiable functions such that $f(2) = 3$, $f'(2) = -1$, $f'(3) = 7$, $g(2) = -5$ and $g'(2) = 2$, find the numbers indicated in problems 40 – 42.

40. $(g-f)'(2)$

41. $(fg)'(2)$

42. $\left(\dfrac{f}{g}\right)'(2)$