Midterm : Next Monday , Oct 29

↳ covers everything up to the end of this lecture .

So far :

Our inventory of derivative rules :

Types of functions

→ Power function : $(x^n)' = nx^{n-1}$

→ Exponentials : $(b^x)' = b^x . \ln b$

$\xrightarrow{\text{special}} (e^x)' = e^x$

→ Trig functions : $(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x = 1 + \tan^2 x$

→ Constant function : $(c)' = 0$

The operations that combine functions

→ Sum or difference : $(f \pm g)' = f' \pm g'$

→ constant multiple : $(\underset{\text{Constant number}}{c} f)' = c f'$

→ division : $\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$ → Quotient Rule

→ product : $\boxed{(f \cdot g)' = f'g + fg'}$ → Product Rule

Clicker Q :(I) What is the slope of the tangent line to the graph of the function $g(t) = \frac{te^t}{\cos t}$ when $\underline{t = 0}$?

derivative at t=0

$g'(0)$

A . 0

C . undefined

[✓ B]. 1

D . -1

(II) What is the equation of the tangent line to the graph of $g(t)$ at $t=0$?

[✓ A.] $y = x$ C. $y = x - 1$

B. $y = x + 1$ D. $y = 1$

(I)

$$g(t) = \frac{\overset{f}{te^t}}{\underset{h}{\cos(t)}}$$

top $f(t) = \underbrace{te^t}_{\text{product}}$ bottom $h(t) = \cos t$

$f'(t) = (t)'e^t + t(e^t)'$ $h'(t) = -\sin t$

$= 1 \cdot e^t + te^t$

$= e^t + te^t$

$$\Rightarrow \underset{\text{Quotient}}{g'(t)} = \frac{f' \cdot h - f \cdot h'}{h^2} = \frac{(e^t + te^t)\cos t \overset{+}{-} te^t(-\sin t)}{(\cos t)^2}$$

$$m_{tan} \text{ at } t=0 : g'(0) = \frac{(\overset{1}{\cancel{e^0}} + \overset{0}{\cancel{0e^0}})\overset{1}{\cancel{\cos 0}} + \overset{0}{\cancel{0e^0 \sin 0}}}{(\underset{1}{\cancel{\cos 0}})^2}$$

$$= \frac{1}{1} = 1 \Rightarrow \boxed{B}$$

(II) equation of the tangent line : at $t=0$ need slope $= m_{tan} = g'(0) = 1$

point $\xrightarrow{t=0} g(0) = \frac{0 \cdot e^0}{\cos 0} = \frac{0}{1} = 0 \Rightarrow (0,0)$

Equation: $y - y_0 = m(x - x_0) \Rightarrow y - 0 = 1(x - 0) \Rightarrow \boxed{y = x}$ $\boxed{A}$

One other way we can combine functions : Composing two function :

$$x \xrightarrow[\text{1st action}]{f} f(x) \xrightarrow[\text{2nd action}]{g} \overset{\text{outside}}{g}(\underbrace{f(x)}_{\text{inside}}) \quad : \quad y = g\circ f(x) = g(f(x))$$

or

$$x \xrightarrow[\text{1st action}]{g} g(x) \xrightarrow[\text{2nd action}]{f} \underbrace{f}_{\text{outside}}(\underbrace{g(x)}_{\text{inside}}) \quad : \quad y = f\circ g(x) = f(g(x))$$

Example :

$$y = \underbrace{\sin}_{\text{2nd : outside}}(\underbrace{x^2}_{\text{inside: 1st}})$$

$$x \xrightarrow{x^2} \boxed{x^2} \xrightarrow{\sin x} \sin(x^2)$$

How to differentiate :

$$y = f(g(x)) \Rightarrow \boxed{y' = f'(g(x)) \cdot g'(x)} \rightarrow \text{Chain Rule}$$

Take the derivative of outside (f) and evaluate it at inside, then multiply by the derivative of inside (g).

Go back to example:

$$y = \sin(x^2) \leadsto y' = \cos \text{▨} \cdot \text{▨}'$$

$$= \underbrace{\sin}_{\text{outside}}(\underbrace{\text{▨}}_{\text{inside: } x^2}) \qquad = \cos(x^2)\cdot(x^2)'$$

$$= \cos x^2 \cdot 2x$$

$f(x) = \sin x \qquad g(x) = x^2$

$f(g(x)) = \sin(x^2)$

$f'(x) = \cos x$

$f'(g(x)) = f'(x^2) = \cos x^2$

What about

$$y = \sin^2 x \leadsto y' = \underbrace{2\,\overset{\text{inside}}{\text{▨}}}_{\text{outside derivative}} \cdot \underbrace{\text{▨}'}_{\text{inside}}$$

outside derivative $(x^2)'$ evaluate at inside ▨

$$= (\sin x)^2$$

$$= (\underbrace{\text{▨}}_{\text{inside : Sin}})^{2 \to \text{outside : } x^2}$$

$$= 2 \sin x \cdot \cos x$$

Examples :

(a) $h(x) = \sqrt{x^2+1}$

(b) $h(x) = \cos(e^x)$

(c) $y = (2x^3+5)^7$

(d) $y = e^{-3x} + \sin(\sqrt{x})$

(a) $\frac{1}{2} \to$ outside

$$h(x) = (x^2+1)^{\frac{1}{2}} = \text{▨}$$

↘ inside

$$h'(x) = \underbrace{\frac{1}{2}\,\text{▨}^{\frac{1}{2}-1}}_{\text{derivative of outside}} \cdot \underbrace{\text{▨}' \rightsquigarrow (x^2+1)'}_{\text{derivative of inside}}$$

$$= \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}} \cdot \cancel{2}x$$

$$= (x^2+1)^{-\frac{1}{2}} \cdot x = \frac{x}{\sqrt{x^2+1}}$$

(b) $$h(x) = \cos(e^x) = \underset{\text{outside}}{\cos}\ \underset{\text{inside}}{\text{▨}}$$

$$h'(x) = -\sin \text{▨} \cdot \text{▨}'$$

$$= -\sin(e^x) \cdot (e^x)'$$

$$= -\sin(e^x) \cdot e^x$$

(c) $$y = (2x^3+5)^7 = \underset{\swarrow\,\text{inside}}{\text{▨}}^{\,7\,\nearrow\,\text{outside}}$$

$$y' = 7\,\text{▨}^{7-1} \cdot (\text{▨})'$$

$$= 7(2x^3+5)^6 \cdot (2x^3+5)'$$

$$= 7(2x^3+5)^6 \cdot 6x^2$$

$$= 42(2x^3+5)^6 \cdot x^2$$

(d) $$y = \underset{\searrow\,\text{outside}}{\boxed{e}}^{\boxed{-3x}\,\text{inside}} + \underset{\searrow\,\text{outside}}{\underbrace{\sin}}\left(\boxed{\sqrt{x}}\right)^{\rightarrow\,\text{inside}}$$

$$y = e^{\text{▨}} + \sin \text{▨}$$

$$y' = \underbrace{e}_{\swarrow\,\text{derivative of outside}}^{\text{▨}} \cdot \underbrace{(\text{▨})'}_{\text{derivative of inside}} + \cos \text{▨} \cdot (\text{▨})'$$

$$= e^{-3x} \cdot (-3x)' + \cos(\sqrt{x}) \cdot \overset{x^{\frac{1}{2}}\ \shortparallel}{(\sqrt{x})'}$$

$$= e^{-3x} \cdot -3 + \cos(\sqrt{x}) \cdot \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= -3e^{-3x} + \frac{1}{2}\cos(\sqrt{x}) \cdot x^{-\frac{1}{2}}$$

→ The derivative of $y = \ln x$

We use the relation between $\ln x$ and e^x with the help of the chain rule and we find $(\ln x)'$:

* Recall : $e^{\ln x} = x$

Differentiate both sides : $\left(e^{\ln x}\right)' = (x)'$

Use the Chain rule to find the derivative of LHS : $e^{\ln x} \cdot (\ln x)' = 1$

but $\boxed{e^{\ln x} = x}$ $\quad x \cdot (\ln x)' = 1$

$$\Rightarrow \boxed{(\ln x)' = \frac{1}{x}}$$

Example :

(a) $y = \ln(5x+2)$

(b) $y = \ln\left(\sqrt{x^2 - 2x}\right)$

(a) $y = \ln(5x+2) = \ln$ ▨

$$\Rightarrow y' = \underbrace{\frac{1}{\text{▨}}}_{\text{derivative of outside: ln}} \cdot \underbrace{\text{▨}'}_{\text{derivative of inside}} = \frac{1}{5x+2} \cdot (5x+2)'$$

$$= \frac{1}{5x+2} \cdot 5$$

$$= \frac{5}{5x+2}$$

(b) $y = \ln\left(\sqrt{x^2-2x}\right)$

$$y = \ln(\text{▨})$$

$$\Rightarrow \quad y' = \frac{1}{\text{▨}} \cdot (\text{▨})'$$

$$= \frac{1}{\sqrt{x^2-2x}} \cdot \underbrace{\left(\sqrt{x^2-2x}\right)'}$$

$$\boxed{= \frac{1}{\sqrt{x^2-2x}} \cdot \frac{1}{2}(x^2-2x)^{-\frac{1}{2}} \cdot (2x-2)}$$

You simplify!

$$\left(\sqrt{x^2-2x}\right)' = \left[(x^2-2x)^{\frac{1}{2}}\right]' = \left(\text{▨}^{\frac{1}{2}}\right)'$$

$$= \underbrace{\frac{1}{2}(x^2-2x)^{\frac{1}{2}-1}}_{\text{outside derivative}} \cdot \underbrace{(2x-2)}_{\text{inside derivative}}$$

2nd way: Simpler

$$y = \ln(x^2-2x)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2-2x)$$

$$\boxed{\text{Recall: } \ln \text{▨}^n = n \ln \text{▨}}$$

$$y' = \frac{1}{2} \cdot \frac{1}{x^2-2x} \cdot (2x-2)$$

This is equal to the boxed expression above

Now you should be able to differentiate all of the following functions (except the crossed ones.)

Derivative Worksheet

1. $f(x) = 4x^5 - 5x^4$

2. $f(x) = e^x \sin x$

3. $f(x) = (x^4 + 3x)^{-1}$

4. $f(x) = 3x^2(x^3 + 1)^7$

5. $f(x) = \cos^4 x - 2x^2$

6. $f(x) = \dfrac{x}{1 + x^2}$

7. $f(x) = \dfrac{x^2 - 1}{x}$

8. $f(x) = (3x^2)(x^{\frac{1}{2}})$

9. $f(x) = \ln(xe^{7x})$

10. $f(x) = \dfrac{2x^4 + 3x^2 - 1}{x^2}$

11. $f(x) = (x^3)\sqrt[5]{2 - x}$

12. $f(x) = 2x - \dfrac{4}{\sqrt{x}}$

13. $f(x) = \dfrac{4(3x - 1)^2}{x^2 + 7^x}$

14. $f(x) = \sqrt{x^2 + 8}$

15. $f(x) = \dfrac{x}{\sqrt{1 - (\ln x)^2}}$

16. $f(x) = \dfrac{6}{(3x^2 - \pi)^4}$

17. $f(x) = \dfrac{(3x^2 - \pi x)^4}{6}$

18. $f(x) = \dfrac{x}{(x^2 + \sqrt{3x})^5}$

19. $f(x) = (xe^x)^\pi$

~~20. $f(x) = \left[\arctan(2x)\right]^{10}$~~

21. $f(x) = (e^{2x} + e)^{\frac{1}{2}}$

22. $f(x) = (x^6 + 1)^5(4x + 7)^3$

23. $f(x) = (7x + \sqrt{x^2 + 3})^6$

24. $f(x) = \dfrac{\frac{1}{x} + \frac{1}{x^2}}{x - 1}$

25. $f(x) = \sqrt[3]{x^2} - \dfrac{1}{\sqrt{x^3}}$

26. $f(x) = \sqrt{\dfrac{2x + 5}{7x - 9}}$

27. $f(x) = \dfrac{\sin x}{\cos x}$

28. $f(x) = e^x(x^2 + 3)(x^3 + 4)$

29. $f(x) = \dfrac{5x^2 - 7x}{x^2 + 2}$

30. $f(x) = \left[\ln(5x^2 + 9)\right]^3$

31. $f(x) = \ln(5x^2 + 9)^3$

32. $f(x) = \cot(6x)$

~~33. $f(x) = \sec^2 x \cdot \tan x$~~

~~34. $f(x) = \arcsin(2^x)$~~

35. $f(x) = \tan(\cos x)$

36. $f(x) = [(x^2 - 1)^5 - x]^3$

~~37. $f(x) = \sec x \cdot \sin(3x)$~~

38. $f(x) = \dfrac{(x - 1)^3}{x(x + 3)^4}$

39. $f(x) = \log_5(3x^2 + 4x)$

If f and g are differentiable functions such that $f(2) = 3$, $f'(2) = -1$, $f'(3) = 7$, $g(2) = -5$ and $g'(2) = 2$, find the numbers indicated in problems 40 – 42

40. $(g - f)'(2)$

41. $(fg)'(2)$

42. $\left(\dfrac{f}{g}\right)'(2)$