Oct 26

Lecture 22 : Midterm Review

Review of Functions

- polynomials → linear functions : $y = mx + b$ or $y - y_0 = m(x - x_0)$

  Domain $= \mathbb{R}$

  → quadratic functions :

  $y = ax^2 + bx + c$ $\quad a > 0$  $\quad a < 0$ 

  vertex : $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
  Coordinates

  → higher degrees

  $y = x^3 + 5x$, $y = 2x^7 + 5x^6$ ...

Solving quadratic equations by quadratic formula :

$$y = ax^2 + bx + c \rightsquigarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

→ $b^2 - 4ac > 0$ → two solutions

→ $b^2 - 4ac = 0$ → One solution

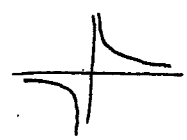
→ $b^2 - 4ac < 0$ → NO solution

- First look for factoring then try quadratic formula.
- Practice factoring , we need it in limits as well .

- Rational functions : $\dfrac{\text{Polynomial}}{\text{polynomial}} = \dfrac{5x^2 + 3x + 1}{10x^4 + x^2}$

  Domain $= \mathbb{R} - \{\text{roots of the denominator}\}$

→ Important example : $y = \frac{1}{x}$



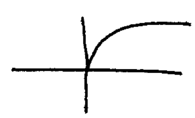
* Remember the graph & its asymptotic behaviour.

- Root functions (square roots) : $\sqrt{\quad}$: Domain : $\quad \geq 0$

  * If square root sits in the denominator we don't consider $= 0$.

→ Important Example : $y = \sqrt{x}$

$x \geq 0$

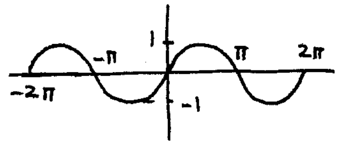


- Trig functions

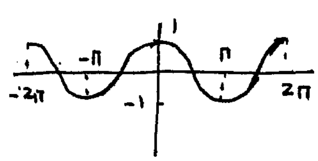
* Domain $= \mathbb{R}$

* Everything in Radian

$y = \sin x$

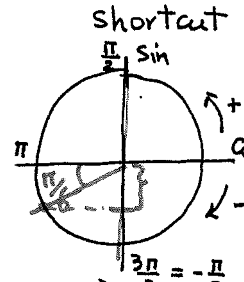


$y = \cos x$



* Range = $[-1, 1]$

* Remember the unit circle shortcut



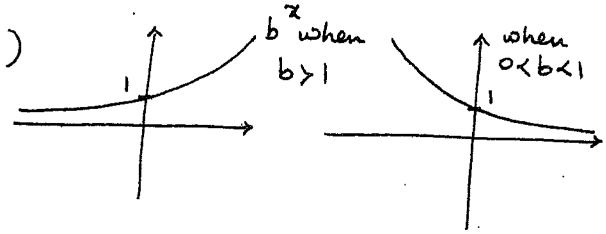
$\frac{7\pi}{6} = \frac{6\pi + \pi}{6}$

$= \pi + \frac{\pi}{6}$

$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$

* Remember Sin & Cos of common angles. $\left(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right)$

- Exponential function $y = b^x$ $(b>0, b\neq 1)$

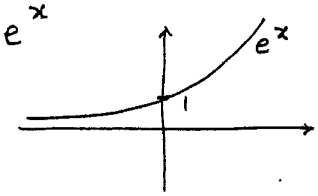


→ Very important : $y = e^x$

Domain : $\mathbb{R}$

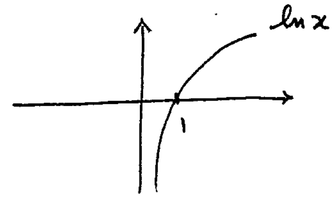
Range : $(0, \infty)$

$e^x \neq 0$ and e^x is NEVER negative.



- logarithm function

$y = \log_b x \xrightarrow{b=e} y = \log_e x = \ln x$



Domain $= (0,\infty)$

Range $= \mathbb{R}$

* Remember these important graphs, they will help you a lot especially when finding V.A. and H.A. of these functions. Also, you should know how to solve exponential & logarithmic equations.

$\boxed{\text{Limits}}$ → Graphically : approach a value on the x-axis, track the graph and find the y-value to which the graph approaches.

↓

Computationally : $\lim\limits_{x\to a} f(x)$

Start with substitution:

I. You get a finite number L → Done ✓

II. $\frac{0}{0}$ → Simplify → factor / common denominator / Conjugate multiplication ($\sqrt{\;}$) → Sub again → Finite number L ✓

III. You get $\frac{1}{0}$, $\frac{\text{Nonzero number}}{0}$ → $\pm\infty$ → Check one side limits to see if 0^+ or 0^- and determine $+\infty$ or $-\infty$ ($\frac{-1}{0^+} = -\infty$)

IV. if $x\to\infty$ or $x\to-\infty$ → Find the dominant term, cancel any common terms and argue if you have $\frac{\text{number}}{\infty}$ → 0, $\frac{\infty}{\text{number}}$ → ∞, $\frac{\text{num}}{\text{num}}$ → finite L, e^x, e^{-x} ...

* In the diagram above

- Case III : Vertical asymptote : $\boxed{\lim\limits_{x\to a} f(x) = \pm\infty}$ → check one-sided limits for ∞ or $-\infty$ ⇒ $x=a$ is V.A. ⇒ vertical tail(s) around "a"

Remember these definitions

- Case IV : Horizontal asymptote : $\boxed{\lim\limits_{x\to\pm\infty} f(x) = b}$ → $y=b$ is HA ⇒ Horizontal tail(s) around $y=b$ ..

$\boxed{\text{* You MUST verify each limit and show limit computation.}}$

$f(x) = \frac{1}{2x} \longrightarrow f(x+h) = \frac{1}{2x} + h$

Derivative → Slope of the tangent line
→ limit definition : $\boxed{f'(x) = \lim_{h \to 0} \frac{f(\boxed{x+h}) - (f(x))}{h}}$ $= \frac{1}{2(x+h)}$

You may need factoring, common denominator or multiplying by the conjugate to find this limit.

- $x+h$ sits for all x given in the function f
- Use brackets when substituting $x+h$
- Use brackets for the 2nd term so that you remember to distribute − sign in all the terms following it.

* Read the question carefully :

→ if $f'(x)$ then use the formula but
→ if $f'(3)$ for example then $x = 3$ so $f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

easier computation.

Derivative Rules

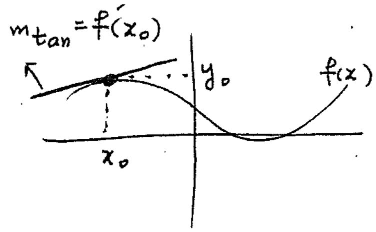
- power rule: $(x^n)' = n x^{n-1}$
- constant function: $(c)' = 0$, $(1)' = 0$, $(\pi)' = 0$, $(e^{\pi})' = 0$
- trigs: $(\cos x)' = -\sin x$, $(\sin x)' = \cos x$, $(\tan x)' = \sec^2 x$
- exp: $(b^x)' = b^x . \ln b$, $(e^x)' = e^x$
- ln : $(\ln x)' = \frac{1}{x}$

Combination of functions

- $(f \pm g)' = f' \pm g'$
- Constant multiple: $(cf(x))' = cf'(x)$, $(5x^2)'$, $(-10e^x)$
- product: $(f.g)' = f'g + fg'$, $(xe^x)'$, $((\ln x)(\sin x))'$
- quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - (fg')}{g^2}$
  put brackets for the 2nd term ⇝ distribute (−)
  $\left(\frac{x+1}{x^3+3}\right)'$, $\left(\frac{\sin x}{x+1}\right)'$
- chain rule
  ↳ functions of functions
  $(f(g(x)))' = f'(g(x)) \cdot g'(x)$
  $(\sin(e^x))'$
  $(e^{\cos x})'$ $(e^{5x})'$
  $\ln(\sin x)$
  $\sqrt{x^2 + x}$
  ⋮

* All these rules can be combined together.
* Can be used multiple times in one function

Equation of the tangent line



$$y - y_0 = m(x - x_0)$$

Comes from $m_{tan} = f'(x)$ evaluated at the tangency point

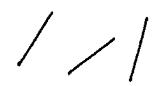
(x_0, y_0) are the coordinates of the tangency point.

Plug x_0 into the original function $f(x)$ and find y_0

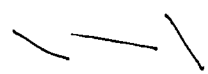
Relation between f and f' graphically

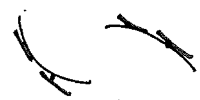
* Always remember f' : slope of the tangent line

$\left(\frac{1}{2x}\right)' = \left(-\frac{1}{4x^2}\right)$ so f' ⊕ → m_{tan} ⊕



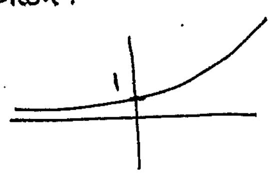
always − f' ⊖ → m_{tan} ⊖





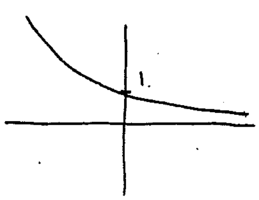
Asymptotes of important function:

$y = e^x$



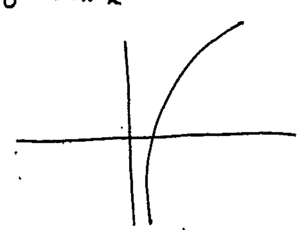
- $\lim_{x \to -\infty} e^x = 0 \Rightarrow y = 0$ H.A. (left)
- $\lim_{x \to \infty} e^x = \infty$
- e^x has NO V.A. : defined everywhere

Similar for e^{-x}



- $\lim_{x \to \infty} e^{-x} = 0 \Rightarrow y = 0$ H.A. (right)
- $\lim_{x \to -\infty} e^{-x} = \infty$
- NO V.A.

$y = \ln x$



- $\ln 0$ is undefined $\Rightarrow \lim_{x \to 0^+} \ln x = -\infty \Rightarrow x = 0$ V.A. (downward)
- $\lim_{x \to \infty} \ln x = \infty$ NO H.A.

Practice Problems

① Evaluate the following : $\sin\left(\frac{5\pi}{6}\right)$, $\cos\left(-\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{4}\right)$, $\sin\left(\frac{11\pi}{3}\right)$, $\cos\left(\frac{7\pi}{6}\right)$

② What limit strategy do you use for each of the following; find each limit

(a) $\lim\limits_{x\to 2} \dfrac{x^2-6x+8}{x-1}$ Substitution $\boxed{a - e} \to c.$

(b) $\lim\limits_{x\to 2} \dfrac{x^2-6x+8}{x-2}$ $\frac{0}{0}$ factor and cancel $\boxed{f - i} \to g$

$\boxed{\text{(c) } \lim\limits_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}}$

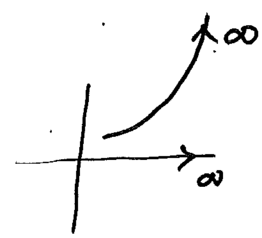
(d) $\lim\limits_{x\to 1} \dfrac{\sqrt{2-x}-1}{x-1}$ Conjugate $\cdot \dfrac{\sqrt{2-x}+1}{\sqrt{2-x}+1}$

(e) $\lim\limits_{x\to 4} \dfrac{|x-4|}{x-4}$ split into $x\to 4^+$, $x\to 4^-$ $\to$ DNE

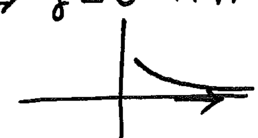
(f) $\lim\limits_{x\to 1} f(x)$ where $f(x) = \begin{cases} x^2-2x+1 & x\le 1 \quad x\to 1^- \\ \ln x & x>1 \quad x\to 1^+ \end{cases}$

(g) $\lim\limits_{x\to \frac{\pi}{2}^+} \dfrac{\sin x}{x-\frac{\pi}{2}}$

(h) $\lim\limits_{x\to \infty} \dfrac{x^2-3x+1}{x+2} \approx \dfrac{x^{\cancel{2}}}{\cancel{x}} = \dfrac{x}{1} = \dfrac{\infty}{1} = \infty$



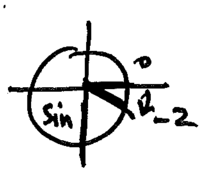
(i) $\lim\limits_{x\to \infty} \dfrac{x+2}{x^2-3x+1} \approx \dfrac{\cancel{x}}{x^{\cancel{2}}} = \dfrac{1}{x} = \dfrac{1}{\infty} = 0 \Rightarrow y=0$ H.A.



A ③ Find V.A. and H.A. of the function $f(x) = \dfrac{x^3-3x^2+1}{2x^3-8x}$

B ④ Find H.A. of $f(x) = \dfrac{1}{e^x+e^{-x}}$

C ⑤ Find V.A. of $f(x) = \dfrac{\sin x}{x+2}$ candidate $x=-2$ $\Rightarrow$ $x=-2$ V.A.



$\lim\limits_{x\to -2^+} \dfrac{\sin x}{x+2} = \dfrac{\sin(-2)}{-2^+ + 2} = \dfrac{\text{nonzero}^-}{0^+} = -\infty$

⑥ Differentiate the following functions.

(a) $e^{x\cos x}$ (b) $\dfrac{\sin x}{3(x+1)}$ (c) $\sqrt{e^{2x} + \cos(x + x^{\frac{1}{3}})}$

(d) $\cos(\ln x)$ (e) $2^{x} \cdot e^{5x^2+x}$

⑦ (I) Find the derivative of the following functions using the limit definition.

(a) $f(x) = \dfrac{2x}{x-1}$ (b) $\dfrac{2}{x}$ (c) $\sqrt{2x-3}$ (d) x^2+x

(II) Find the derivatives of (a)-(d) using another method of your choice.

(III) Find the equation of the tangent line to the graph of function (b) at $x=-1$ and sketch the graph & its tangent line.

(IV) T/F : function (a) is always increasing.
function (b) is always decreasing.
function (d) is always increasing.

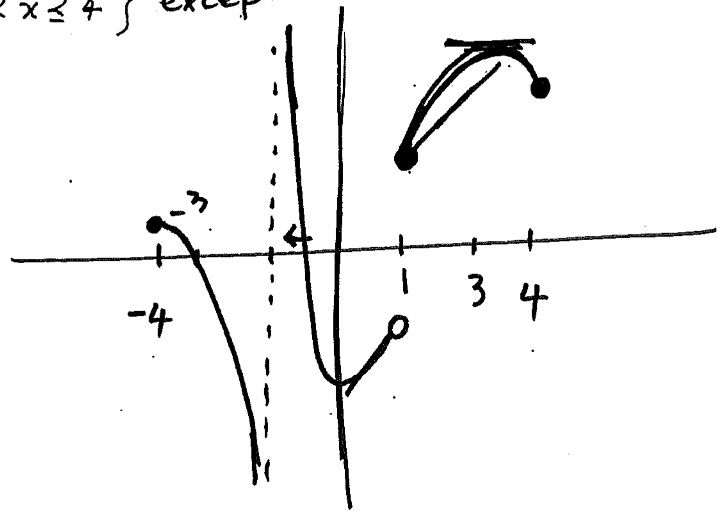
⑧ Find all x values where the following function has a horizontal tangent line.

$$f(x) = x^3 - \tfrac{1}{2}x^2 - 2x + 1800$$

⑨ Sketch a function satisfying the following conditions.

- Domain of f is $\{x \in \mathbb{R} ; -4 \le x \le 4\}$ except $x=-1$
- $f(x)$ has a V.A. at $x=-1$
- $\lim\limits_{x \to -1^+} f(x) = \infty$
- $\lim\limits_{x \to 1} f(x)$ does NOT exist
- $f'(3) = 0$
- $f'(-3) < 0$

$m_{tan} < 0$



(c) $$\lim_{x\to 3} \frac{\frac{1\cdot 3}{x\cdot 3} - \frac{1\cdot x}{3\cdot x}}{x-3} = \frac{\frac{1}{3}-\frac{1}{3}}{3-3} = \frac{0}{0}$$

$$\lim_{x\to 3} \frac{\frac{3}{3x} - \frac{x}{3x}}{x-3} = \lim_{x\to 3} \frac{\frac{3-x}{3x}}{x-3}$$

$$3 - x = -(-3) - x$$

$$3 = -(-3)$$

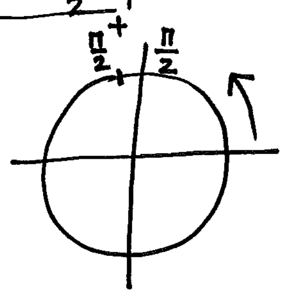
$$= -1(-3+x)$$

$$\frac{-\cancel{(x-3)}^{1}}{} \cdot \frac{1}{\cancel{(x-3)}_{1}}$$

$$= \lim_{x\to 3} \frac{\overset{-\cancel{(x-3)}^{1}}{3-x}}{3x} \cdot \frac{1}{\cancel{x-3}_{1}}$$

$$= \lim_{x\to 3} \frac{-1}{3x} = \frac{-1}{9}$$

(g) $$\lim_{\boxed{x\to\frac{\pi}{2}^{+}}} \frac{\sin x}{x-\frac{\pi}{2}} = \frac{\sin\frac{\pi}{2}}{\underbrace{\frac{\pi}{2}^{+} - \frac{\pi}{2}}_{0^{+}}} = \frac{1}{0^{+}} = +\infty$$



$$\Rightarrow \quad x = \frac{\pi}{2} \quad \text{V. A.}$$

④ $f(x) = \dfrac{1}{e^{x}+e^{-x}}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{e^{x}+e^{-x}} = \frac{1}{\infty+0} = \frac{1}{\infty} = 0$$

$x \to \infty \;:\; e^{x} \to \infty$

$x \to \infty \;:\; e^{-x} \to 0$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} \frac{1}{e^{x}+e^{-x}} = \frac{1}{0+\infty} = \frac{1}{\infty} = 0$$

$x \to -\infty \;:\; e^{x} \to 0$

$x \to -\infty \;:\; e^{-x} \to \infty$ $\Longrightarrow$ $y = 0$ **H.A.**