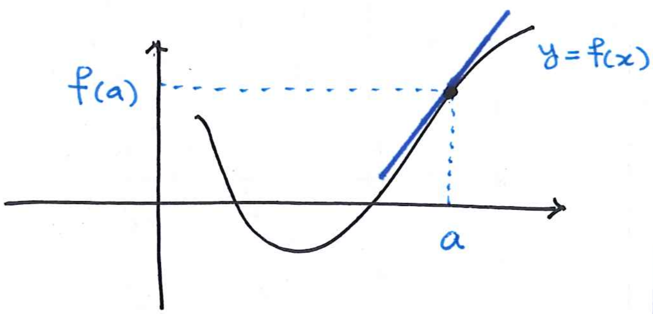
Last class : Derivative of $f(x)$

Slope of the tangent line to the graph of $f(x)$ at any x

limit definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

→ Equation of the tangent line to the graph of $y = f(x)$ at $x = a$



Recall:

* To write the equation of a line we need two things

Slope m | a point on the line (x_0, y_0)

then $y - y_0 = m(x - x_0)$

Since this line is tangent we can find its slope m_{tan} by evaluating the derivative at "a"

so $m_{tan} = f'(a)$

and the point on the line is the tangency point where $f(x)$ and the line touch : $(a, f(a))$

so the equation of the tangent line is :

$$\boxed{y - f(a) = f'(a)(x - a)}$$ → important

⊛ Note that the fact that $(a, f(a))$ is both on the line and on the function help us to get info about line and function both.

Derivatives we computed so far

$f(x) = x^2 \xrightarrow{x^{2-1}} f'(x) = 2x$

$f(x) = x^3 \xrightarrow{x^{3-1}} f'(x) = 3x^2$

Now you guess

$f(x) = x^4 \xrightarrow{x^{4}} f'(x) = 4x^3$

$f(x) = x^9 \rightsquigarrow f'(x) = 9x^8$

$\vdots$

$f(x) = x^{100} \rightsquigarrow f'(x) = 100x^{99}$

In general, for any real number n :

$$f(x) = x^n \rightsquigarrow \boxed{f'(x) = n x^{n-1}}$$ → Power Rule of differentiation

The rigorous mathematical way to show that power rule is really true is by using the limit definition of the derivative. One can verify that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

algebra steps:

$$= nx^{n-1}$$

we did this when $n=2$ & $n=3$.

Example 1. Use power rule to find $f'(x)$.

a) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2} \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}} \xRightarrow[x=4]{\text{for example}} f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

b) $f(x) = \frac{1}{x} = x^{-1}$ ⇝ Recall: $\frac{1}{x^n} = x^{-n}$

$f'(x) = -1\, x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

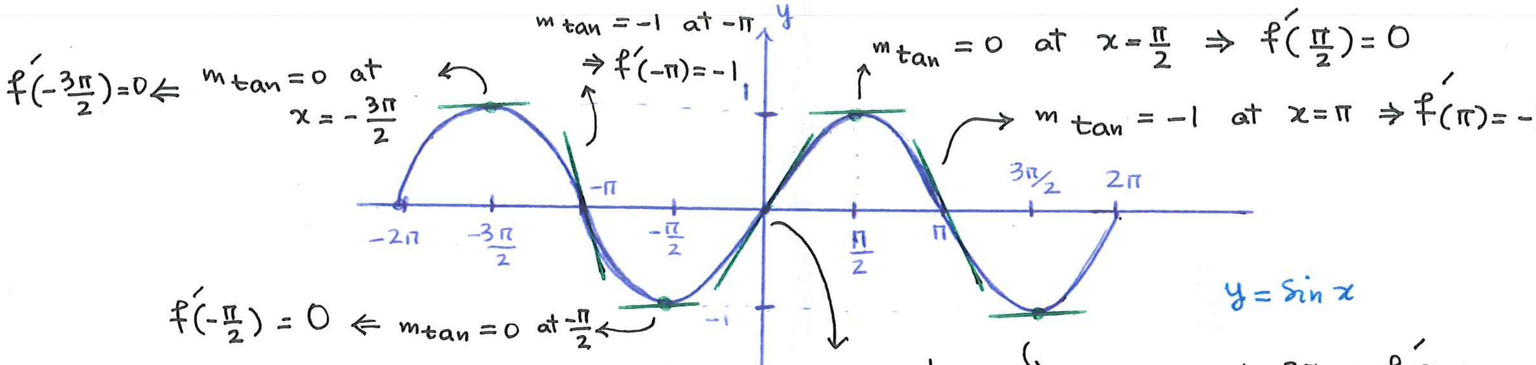
Recall: $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

c) $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}} \Rightarrow f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

e) $f(x) = x^{\pi} \Rightarrow f'(x) = \pi x^{\pi - 1}$

What is the rule for the trig functions $\sin x$ and $\cos x$.

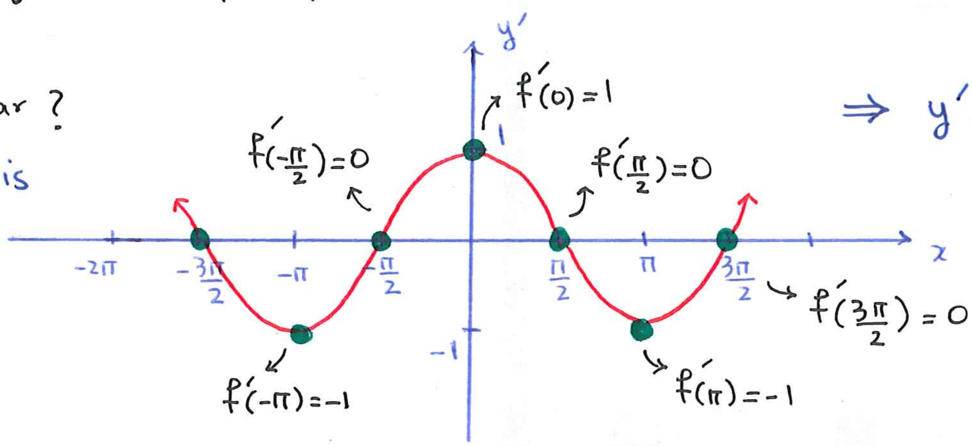
Recall the graph of $y = \sin x$



We track m_{tan} at different points and we'll get a graph for $f'(x)$:

Look familiar?

Yes, this is the graph of $\cos x$



$\Rightarrow y' = \cos x$

$$\boxed{f(x) = \sin x \quad \rightsquigarrow \quad f'(x) = \cos x}$$

In fact, one can rigorously verify this by apply the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

bunch of algebra with trig identities, etc.

$$= \quad \vdots$$

$$= \cos x$$

We can do a similar process for

$$\boxed{f(x) = \cos x \rightsquigarrow f'(x) = -\sin x}$$

and

$$\boxed{f(x) = \tan x \rightsquigarrow f'(x) = 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}}$$

Now let's go to the derivative of another family of functions:

Exponential functions:

$$f(x) = 2^x \rightsquigarrow f'(x) = 2^x \cdot \ln 2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$

$$= \ldots$$

bunch of algebra

$$= 2^x \cdot \ln 2$$

power rule applies? NO

$$\cancel{f'(x) = x\, 2^{x-1}}$$

2^x is an exponential function not a power function so power rule does NOT apply.

Similarly:

$$f(x) = 3^x \rightsquigarrow f'(x) = 3^x \cdot \ln 3$$

$$\boxed{f(x) = e^x \rightsquigarrow f'(x) = e^x \cdot \underbrace{\ln e}_{1} = e^x}$$

→ Important property of e^x: $f(x) = f'(x)$

In general:

$$\boxed{f(x) = b^x \rightsquigarrow f'(x) = b^x \cdot \ln b \quad b > 0}$$

Compare:

$f(x) = x^2 \Rightarrow f'(x) = 2x$

&

$f(x) = 2^x \Rightarrow f'(x) = 2^x \cdot \ln 2$

$f(x) = x^\pi \Rightarrow f'(x) = \pi x^{\pi - 1}$

&

$f(x) = \pi^x \Rightarrow f'(x) = \pi^x \cdot \ln \pi$

Clicker Q . What is the equation of the tangent line to $f(x) = e^x$ at $x = 0$?

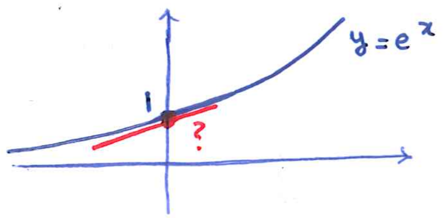
A. $y = x$

B. $y = x + 1$

C. $y = x - 1$

D. $y = ex + 1$

$f(x) = e^x$ at $x = 0$



We need slope at $x = 0$ and the tangency point

m_{tan} at $x = 0$: $f'(0)$

tangency point : $x = 0$, $y = e^0 = 1$ $\Rightarrow (0,1)$

$f'(x) = e^x \xrightarrow{x=0} f'(0) = e^0 = 1 = m_{tan}$

$\Rightarrow y - y_0 = m(x - x_0) \Rightarrow y - 1 = 1(x - 0) \Rightarrow \boxed{y = x + 1}$

Practice . Find the equation of the tangent line to $f(x) = \cos x$ at $x = \frac{\pi}{2}$. Sketch $f(x)$ and its tangent line at $x = \frac{\pi}{2}$.

Clicker Q . $f(x) = x^3 + \sin x$, $f'(x) = ?$

A. $3x^2 + \cos x$

B. $3x^2 + \sin x$

C. $x^3 + \cos x$

D. $3x^2 - \sin x$

$(x^3)' = 3x^2$ and $(\sin x)' = \cos x$ $\Rightarrow$ $(x^3 + \sin x)' = 3x^2 + \cos x$

$\boxed{A}$

In general :

$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$(f(x) - g(x))' = f'(x) - g'(x)$$

Clicker Q : $f(x) = 5$, $f'(x) = ?$

A. 5   C. undefined

B. 0   D. Don't Know

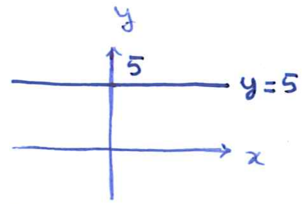
Another way to verify:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5-5}{h} = 0$$

$f(x) = 5$

Constant function



$\Rightarrow m_{tan} = 0$ for all x

$\Rightarrow f'(x) = 0$

In general

$$f(x) = \underbrace{C}_{\text{Constant}} \rightsquigarrow f'(x) = 0$$

Clicker Q : $f(x) = 5 \sin x$, $f'(x) = ?$ $= 5 \cos x$

A. 5   C. $5 \cos x$

B. 0   D. $\cos x$

In general

$$f(x) = \underbrace{c}_{\text{Constant multiple}} \cdot g(x) \rightsquigarrow f'(x) = c\, g'(x)$$

Constant multiple stays in the derivative.

Practice . Find $f'(x)$.

(a) $f(x) = 4x^2 - 3x + 2$

(b) $f(x) = 5 \cos x + 2^x + 5$

(c) $f(x) = 3e^x + \sqrt[3]{x} + \frac{3}{x^2} - 10$

(d) $f(x) = x^{\pi} - \pi^x - \frac{x}{3} + \frac{3}{x}$

(a) $f(x) = 4x^2 - 3x + 2$

$$\left.\begin{aligned} (4x^2)' &= 4(x^2)' = 4\cdot 2x = 8x \\ (3x)' &= 3(x)' = 3\cdot 1 = 3 \\ (2)' &= 0 \end{aligned}\right\} \Rightarrow f'(x) = 8x - 3$$

(b) $f(x) = 5\cos x + 2^x + 5$

$$\left.\begin{aligned} (5\cos x)' &= 5(\cos x)' = 5(-\sin x) \\ (2^x)' &= 2^x \cdot \ln 2 \\ (5)' &= 0 \end{aligned}\right\} \Rightarrow f'(x) = -5\sin x + 2^x \cdot \ln 2$$

(c) $f(x) = 3e^x + \sqrt[3]{x} + \dfrac{3}{x^2} - 10$

$$\left.\begin{aligned} (3e^x)' &= 3(e^x)' = 3e^x \\ (\sqrt[3]{x})' &= \left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \\ \left(\frac{3}{x^2}\right)' &= (3x^{-2})' = 3(x^{-2})' = 3\cdot(-2)x^{-3} \\ &= \frac{-6}{x^3} \\ (10)' &= 0 \end{aligned}\right\} \Rightarrow f'(x) = 3e^x + \frac{1}{3\sqrt[3]{x^2}} - \frac{6}{x^3}$$

(d) $f(x) = x^{\pi} - \pi^x - \dfrac{x}{3} + \dfrac{3}{x}$

$$\left.\begin{aligned} (x^{\pi})' &= \pi x^{\pi - 1} \\ (\pi^x) &= \pi^x \cdot \ln \pi \\ \left(\frac{x}{3}\right)' &= \left(\frac{1}{3}x\right)' = \frac{1}{3}(x)' = \frac{1}{3} \\ \left(\frac{3}{x}\right)' &= (3x^{-1})' = 3(x^{-1})' = 3\cdot(-1)x^{-2} = -3\frac{1}{x^2} \end{aligned}\right\} \Rightarrow f'(x) = \pi x^{\pi-1} - \pi^x \cdot \ln \pi - \frac{1}{3} - \frac{3}{x^2}$$